

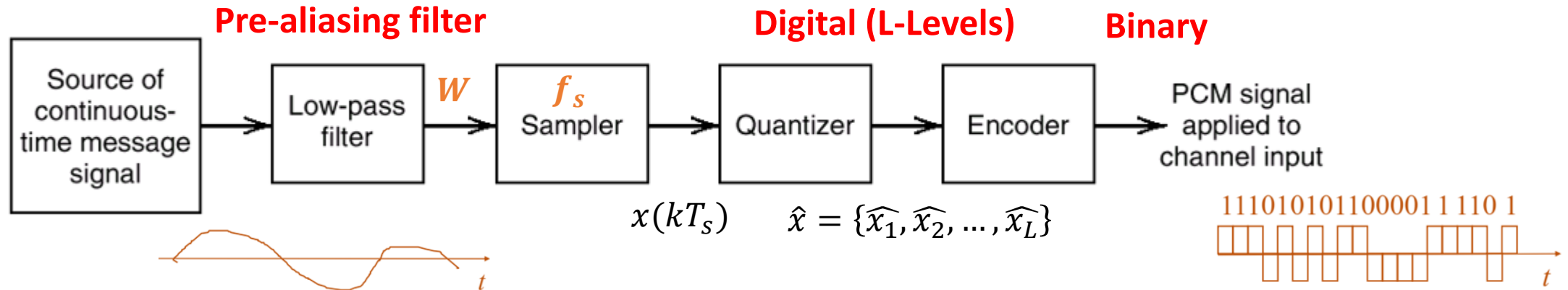
Sampling Theorem

Lecture Outline

- In this this, and the next few lecture, we will address the subject of **pulse code modulation**, where an analog source can be converted into a digital waveform via **sampling**, **quantization**, and **binary encoding**.
- This lecture focuses on Ideal Sampling and the Sampling Theorem.
- The phenomenon of aliasing is explained in detail.
- In the next lecture, we will present two other sampling techniques, which are natural and flat-topped sampling.

Pulse Code Modulation

- Sources are of two types; **analog and digital**. For an analog source, the input transducer is used to convert the physical message generated by the source into a time-varying electrical signal called the *message signal* (like the human voice). This is a continuous time continuous amplitude signal.
- An analog source can be converted into digital via sampling, quantization, and encoding. This process is called **pulse code modulation**



- **Sampler:** If W is the highest frequency component in a signal, then the sampling rate required to reconstruct the message from its samples should follow the Nyquist Rate where $f_s > 2W$.
- The output of the sampler is a continuous amplitude discrete time signal.
- **Quantizer:** Converts the continuous amplitude samples $x(kT_s)$ into **discrete** level samples $\hat{x}(kT_s)$ taken from a finite set of L possible values $\hat{x} = \{\hat{x}_1, \hat{x}_2, \dots, \hat{x}_L\}$.
- **Binary Encoder:** Each quantized level is represented by $r = \log_2 L$ binary digits

Sampling Techniques

- **Sampling:** is the process by which a continuous time continuous amplitude signal is converted into a discrete time continuous amplitude signal.

There are three types of sampling:

- Ideal sampling: To be presented in this lecture
- Natural sampling: To be discussed in the next lecture
- Flat-topped sampling (sample and hold): To be discussed in the next lecture.

Ideal Sampling: The Periodic Train of Impulses

- **Periodic Signals:** A periodic signal $g(t)$ is expanded in the complex Fourier series form as:

- $g(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t} \Rightarrow \mathfrak{F}\{g(t)\} = \sum_{n=-\infty}^{\infty} C_n \delta(f - nf_0)$

Example: Consider the following train of impulses $g(t) = \sum_{m=-\infty}^{\infty} \delta(t - mT_0)$

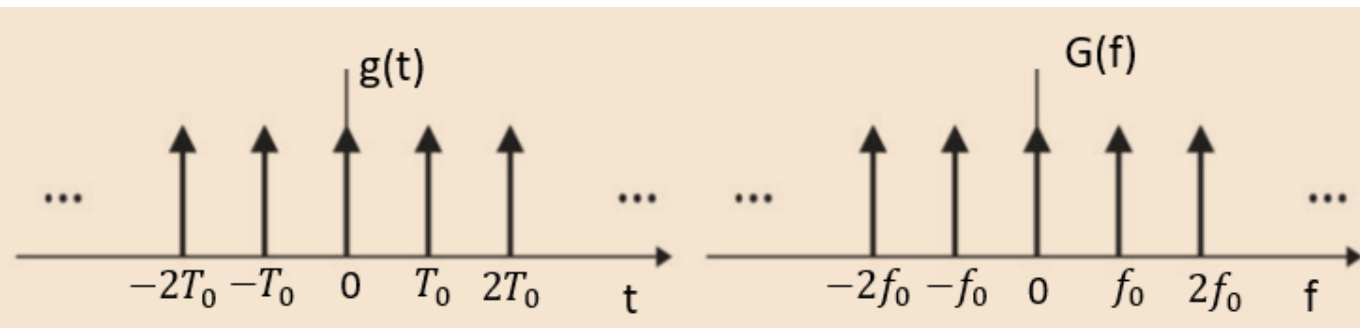
Solution: The Fourier coefficients are obtained by integrating over one period of $g(t)$.

- $C_n = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} g(t) e^{-jn\omega_0 t} dt = \frac{1}{T_0} = f_0$; Note that the sifting property has been used.

- Therefore, the complex Fourier series of $g(t)$ is

- $g(t) = \frac{1}{T_0} \sum_{n=-\infty}^{\infty} e^{jn\omega_0 t}; \Rightarrow \mathfrak{F}\{g(t)\} = \frac{1}{T_0} \sum_{n=-\infty}^{\infty} \mathfrak{F}\{e^{jn\omega_0 t}\} = \frac{1}{T_0} \sum_{n=-\infty}^{\infty} \delta(f - nf_0)$

- $\mathfrak{F} \sum_{m=-\infty}^{\infty} \delta(t - mT_0) = \frac{1}{T_0} \sum_{n=-\infty}^{\infty} \delta(f - nf_0)$.

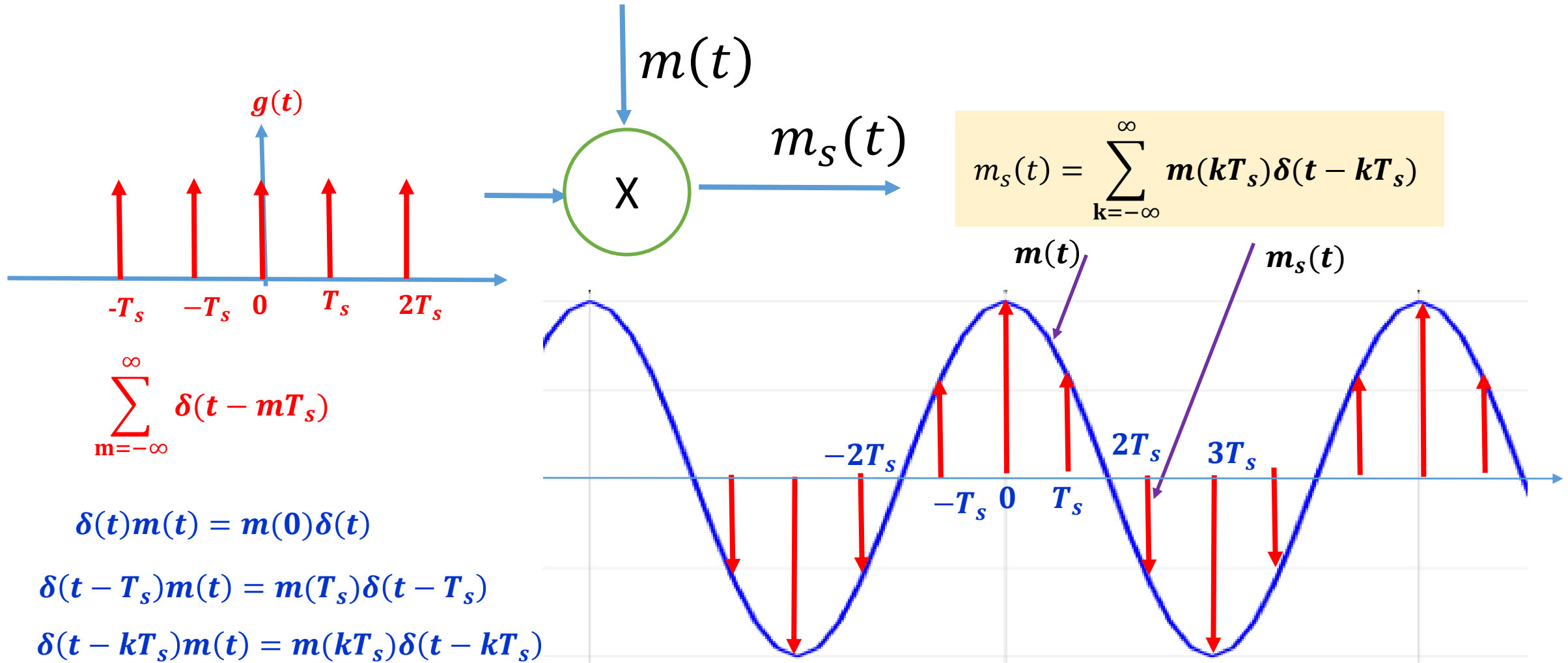


Remark 1: Note that the signal is periodic in the time domain and its Fourier transform is periodic in the frequency domain.

Remark 2: This sequence will be found useful when the sampling theorem is considered later in this lecture.

Ideal Sampling

- **Ideal Sampling:** The message $m(t)$, with Fourier transform $M(f)$, which is band-limited to W Hz, is multiplied by a periodic sequence of ideal impulses with period T_s to produce the sampled signal $m_s(t)$.



Ideal Sampling

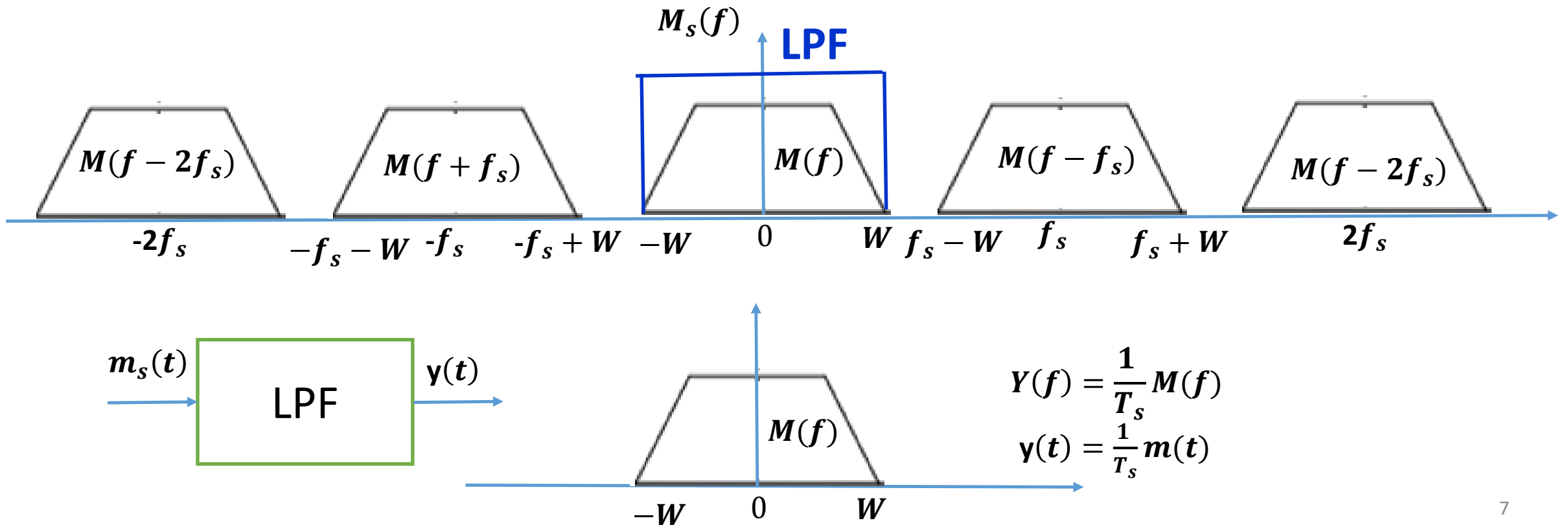
- $m_s(t) = m(t)g(t) = m(t) \sum_{m=-\infty}^{\infty} \delta(t - mT_s) = \sum_{k=-\infty}^{\infty} m(kT_s)\delta(t - kT_s)$
- Recall the Fourier transform pair: $G(f) = \mathfrak{F}(g(t)) = \frac{1}{T_0} \sum_{n=-\infty}^{\infty} \delta(f - nf_0)$
- Hence, $m_s(t) = m(t) \frac{1}{T_0} \sum_{n=-\infty}^{\infty} e^{jn\omega_0 t}$; Product of two functions in the time domain.
- The Fourier transform of $m_s(t)$ is the convolution of $M(f)$ and $G(f)$

$$M_s(f) = M(f) * G(f) = M(f) * \frac{1}{T_s} \sum_{n=-\infty}^{\infty} \delta(f - nf_s)$$
$$M_s(f) = \frac{1}{T_s} \sum_{n=-\infty}^{\infty} M(f - nf_s)$$

Ideal Sampling: $f_s > 2W$

$$M_s(f) = \frac{1}{T_s} \sum_{n=-\infty}^{\infty} M(f - nf_s)$$

- Let $M(f)$ be as given in the figure. When $f_s > 2W$, $M_s(f)$ will look like



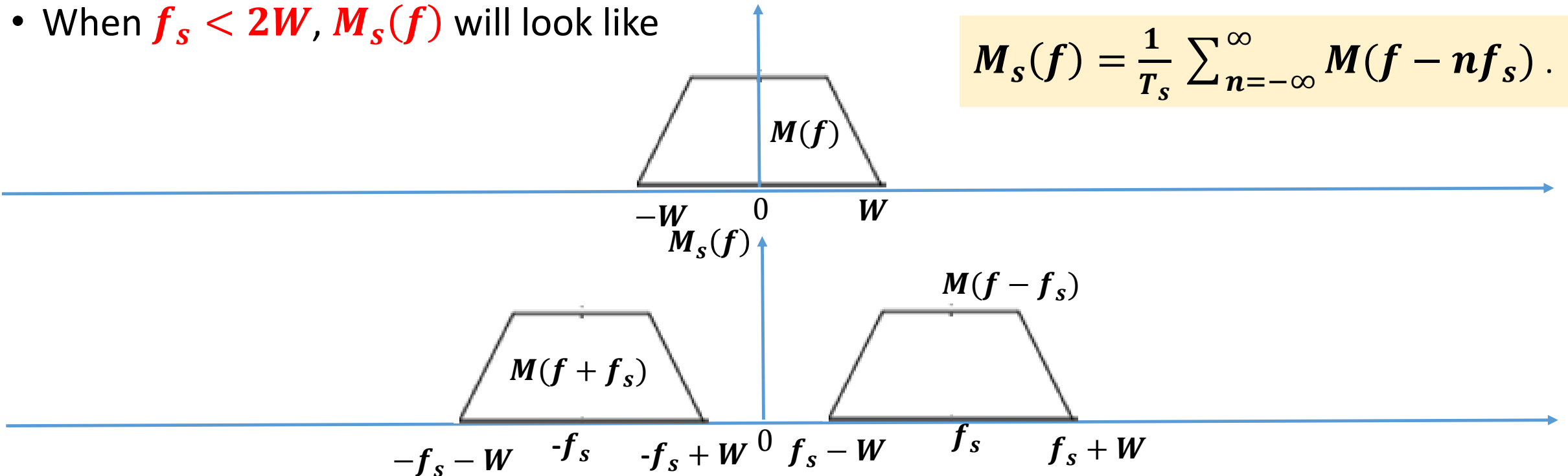
The Sampling Theorem

- A bandlimited signal with no frequency components above W Hz can be recovered uniquely from its samples taken every T_s seconds provided that $f_s \geq 2W$, where $f_s = 1/T_s$ is the sampling rate in samples/sec.
- The message $m(t)$ can be recovered from $m_s(t)$ using an ideal LPF with bandwidth W .
- The Sampling frequency $f_s = 2W$, is called the Nyquist rate. It represents the minimum rate at which a signal must be sampled in order to reconstruct it from its samples without distortion.
- When the sampling rate is less than the Nyquist rate, a distortion type of noise called **Aliasing** results.

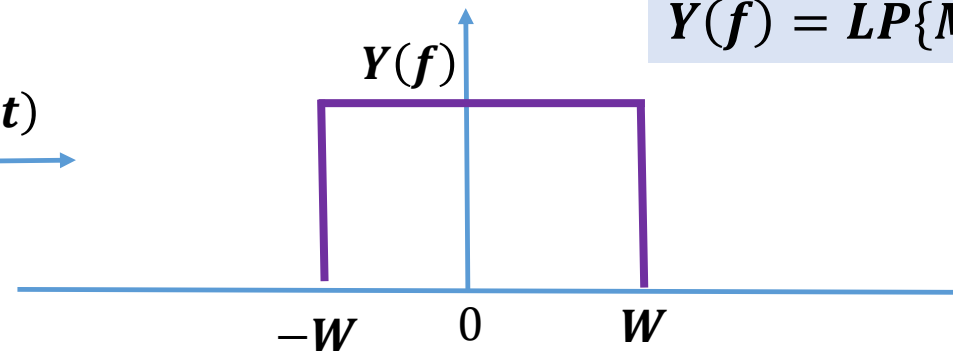
Sampling Theorem and Aliasing

- Let $M(f)$ be the Fourier transform of the message $m(t)$.
- When $f_s < 2W$, $M_s(f)$ will look like

$$M_s(f) = \frac{1}{T_s} \sum_{n=-\infty}^{\infty} M(f - nf_s)$$



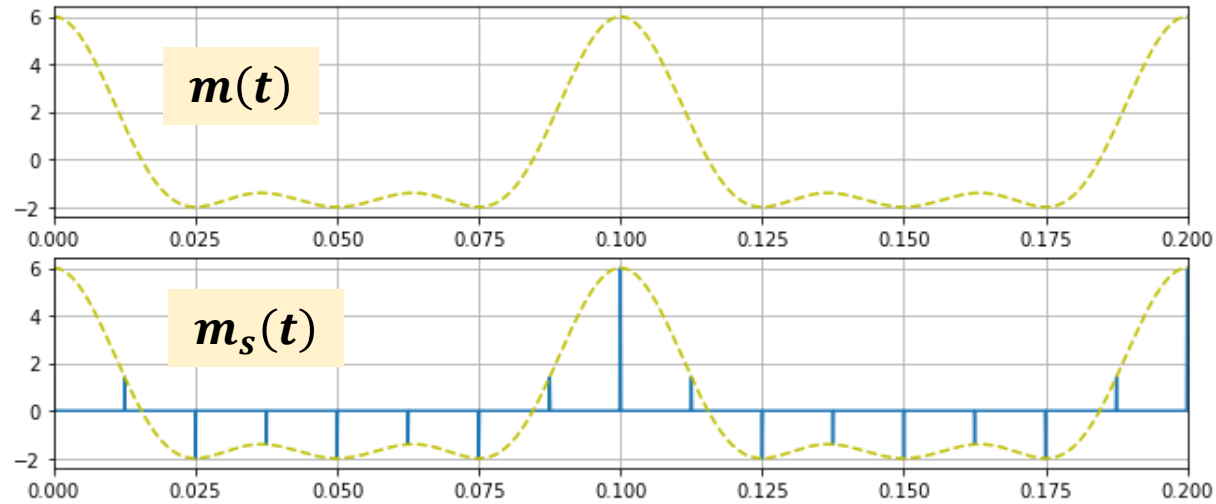
$$Y(f) = LP\{M(f) + M(f - f_s) + M(f + f_s)\}$$



$$Y(f) \neq \frac{1}{T_s} M(f)$$

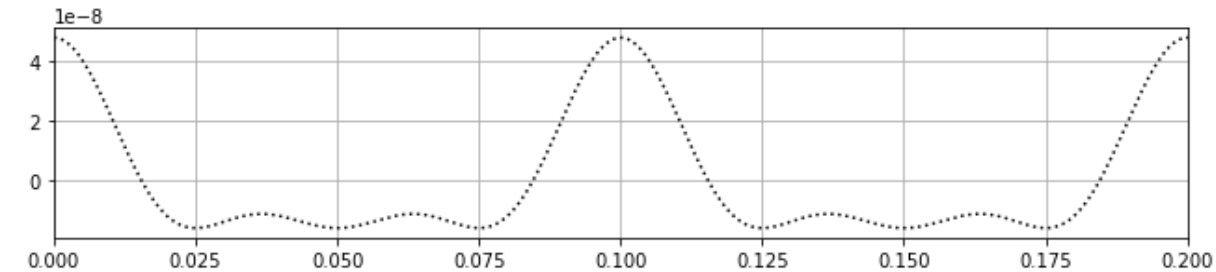
$$y(t) \neq \frac{1}{T_s} m(t)$$

Sampling Theorem: Example $f_s > 2W$

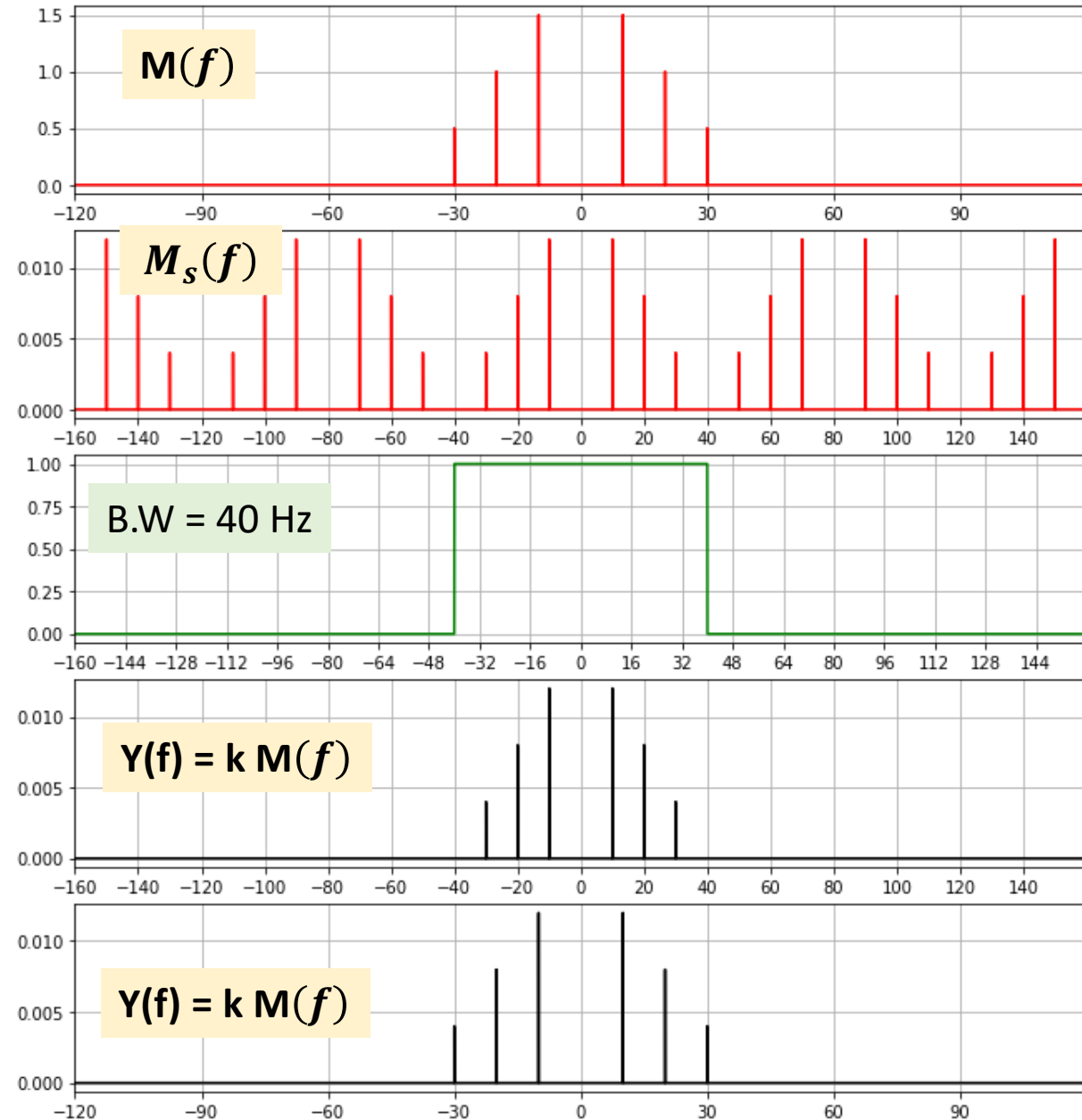


$$m(t) = 3\cos 2\pi(10)t + 2\cos 2\pi(20)t + \cos 2\pi(30)t$$

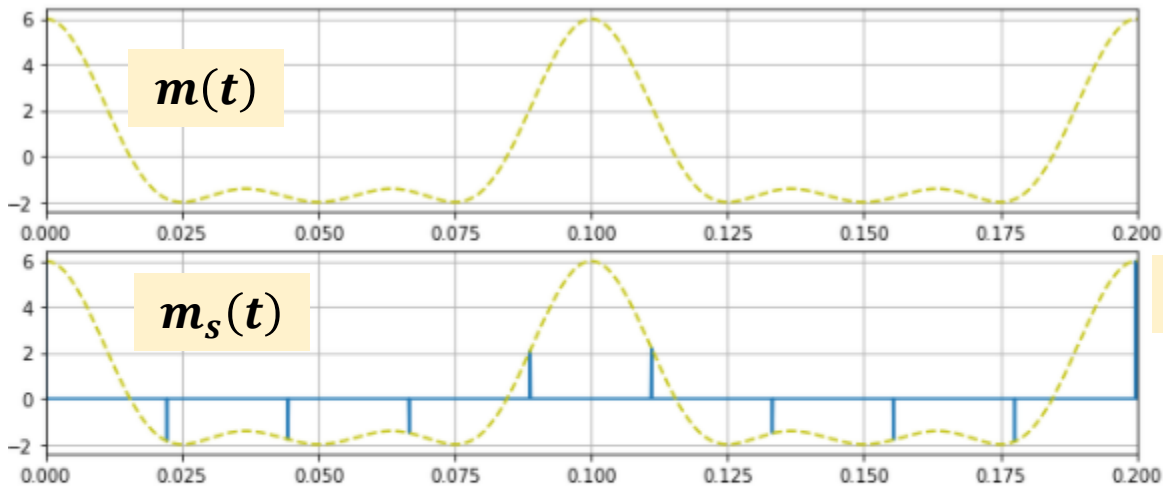
$$f_s = 80; T_s = 0.0125$$



- Since the sampling rate is greater than the Nyquist rate, the original signal is recovered without distortion.
- Output contains the message frequencies: 10, 20, 30



Sampling Theorem and Aliasing : $f_s < 2W$

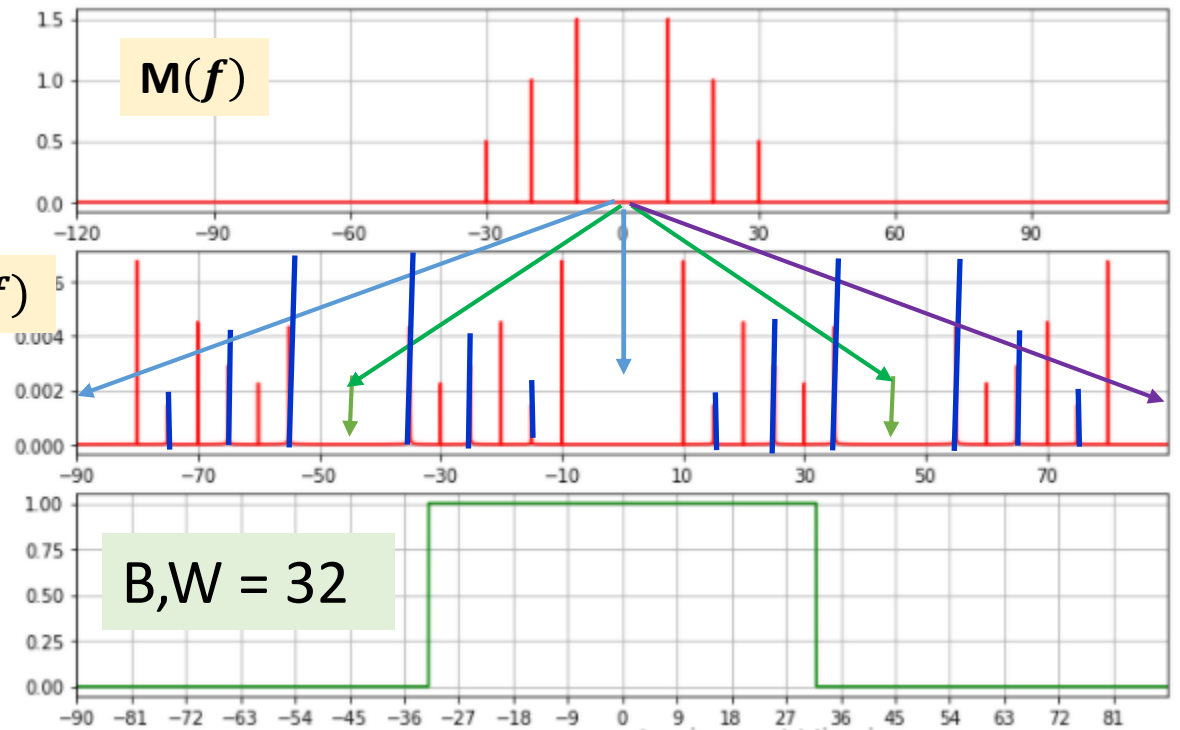


$$m(t) = 3\cos 2\pi(10)t + 2\cos 2\pi(20)t + \cos 2\pi(30)t$$

$$f_s = 45; T_s = 1/45 = 0.022$$

1e-8

$M_s(f)$

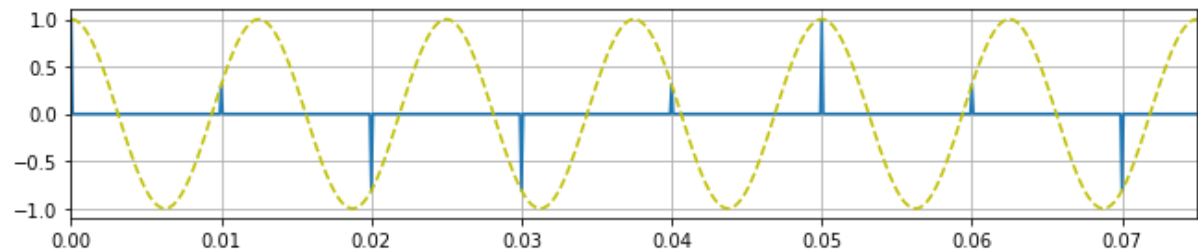
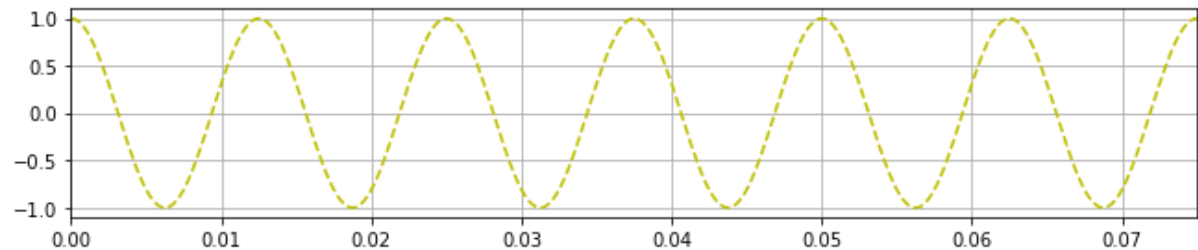


$$M_s(f) = \frac{1}{T_s} \{ M(f) + M(f - f_s) + M(f + f_s) + \dots \}$$

Output contains the message frequencies: 10, 20, 30 Hz. In addition to aliasing frequencies within message bandwidth $(45 - 20) = 25$ and $(45 - 30) = 15$

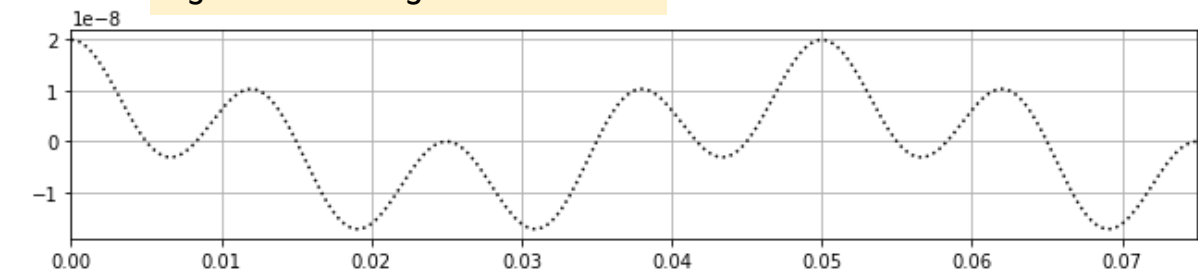
$$y(t) = k \{ 3\cos 2\pi(10)t + 2\cos 2\pi(20)t + \cos 2\pi(30)t + 2\cos 2\pi(25)t + \cos 2\pi(15)t \}$$

Sampling Theorem and Aliasing : Example $f_s < 2W$

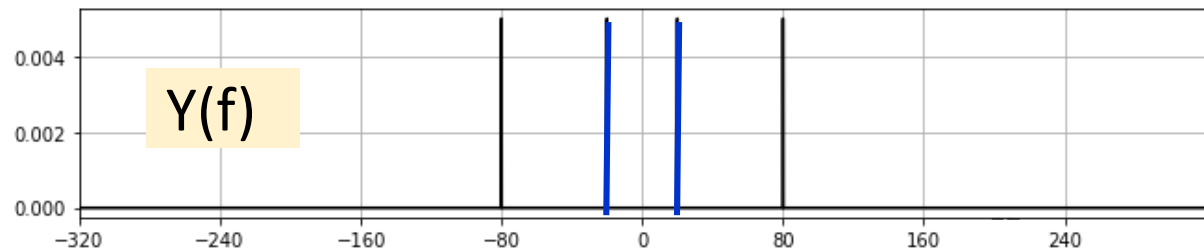
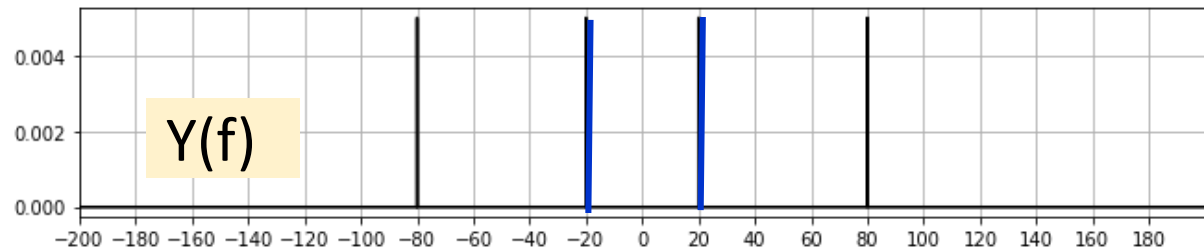
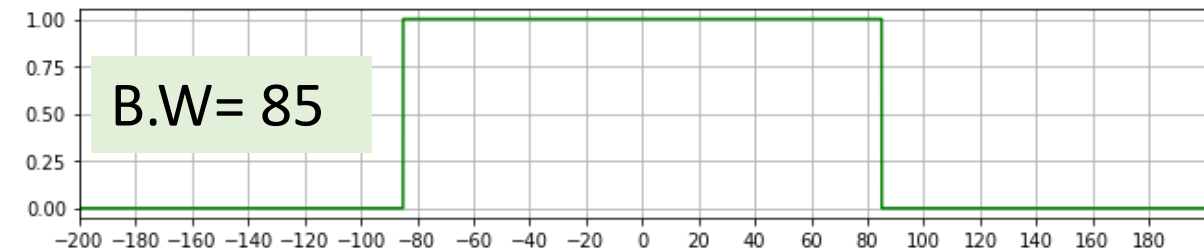
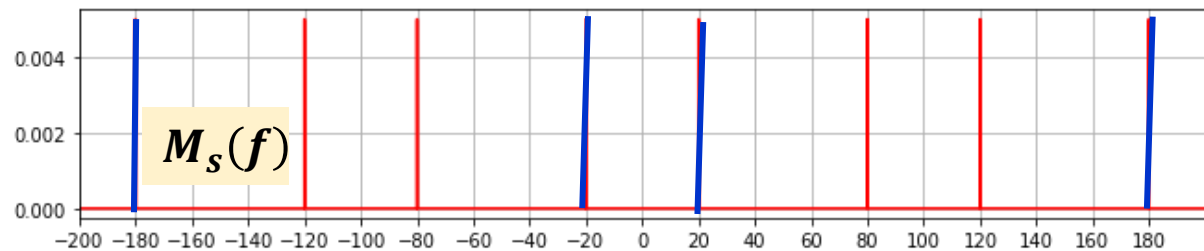
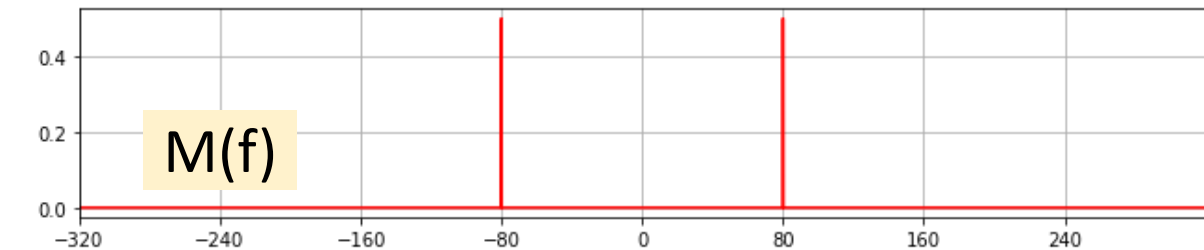


$$m(t) = \cos 2\pi(80)t$$

$$f_s = 100; T_s = 0.01 =$$



$$y(t) = k\cos 2\pi(80)t + k\cos 2\pi(20)t$$



Natural Sampling

Lecture Outline

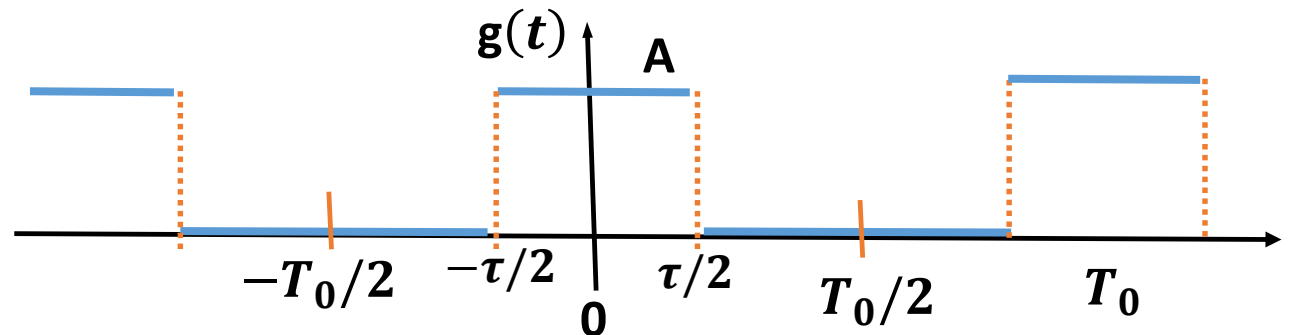
- **Sampling:** is the process by which a continuous time continuous amplitude signal is converted into a discrete time continuous amplitude signal.
- **There are three types of sampling:** Ideal sampling, natural sampling, and flat-topped sampling.
- Ideal sampling, the sampling theorem, and the phenomenon of aliasing were presented in the previous lecture.
- This lecture focuses on natural sampling and the sampling theorem.
- In the next lecture, we consider
 - Flat-topped sampling (sample and hold).
 - Time division multiplexing (TDM)

The Periodic Train of Rectangular Pulses: Fourier Series

- **Example:** Find the trigonometric Fourier series of the periodic rectangular signal defined over one period T_0 as:

$$g(t) = \begin{cases} +A, & -\tau/2 \leq t \leq \tau/2 \\ 0, & \text{otherwise} \end{cases}$$

- **Solution:** The FS is given as $g(t) \sim a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)$
- $a_0 = \frac{1}{T_0} \int_{-\tau/2}^{\tau/2} g(t) dt \Rightarrow a_0 = A\tau/T_0$; τ/T_0 is called the duty cycle of the pulse train.
- $b_n = \frac{2}{T_0} \int_{-\tau/2}^{\tau/2} g(t) \sin(\frac{2\pi n}{T_0} t) dt = 0$; $g(t)$ is an even function of t
- $a_n = \frac{2}{T_0} \int_{-\tau/2}^{\tau/2} g(t) \cos(\frac{2\pi n}{T_0} t) dt = \frac{4}{T_0} \int_0^{\tau/2} A \cos(\frac{2\pi n}{T_0} t) dt \Rightarrow a_n = \frac{2A}{n\pi} \sin(\frac{n\pi\tau}{T_0})$
- $a_n = 0$ when $n = \frac{T_0}{\tau}, \frac{2T_0}{\tau}, \frac{3T_0}{\tau}, \dots$
- This is demonstrated on the next slide



The Periodic Train of Rectangular Pulses: Time and Frequency

The Fourier series of the pulse train is: $g(t) \sim a_0 + \sum_{n=1}^{\infty} (a_n \cos 2\pi n f_0 t)$

The Fourier transform is: $G(f) = a_0 \delta(f) + \sum_{n=1}^{\infty} \frac{a_n}{2} [\delta(f - n f_0) + \delta(f + n f_0)]$

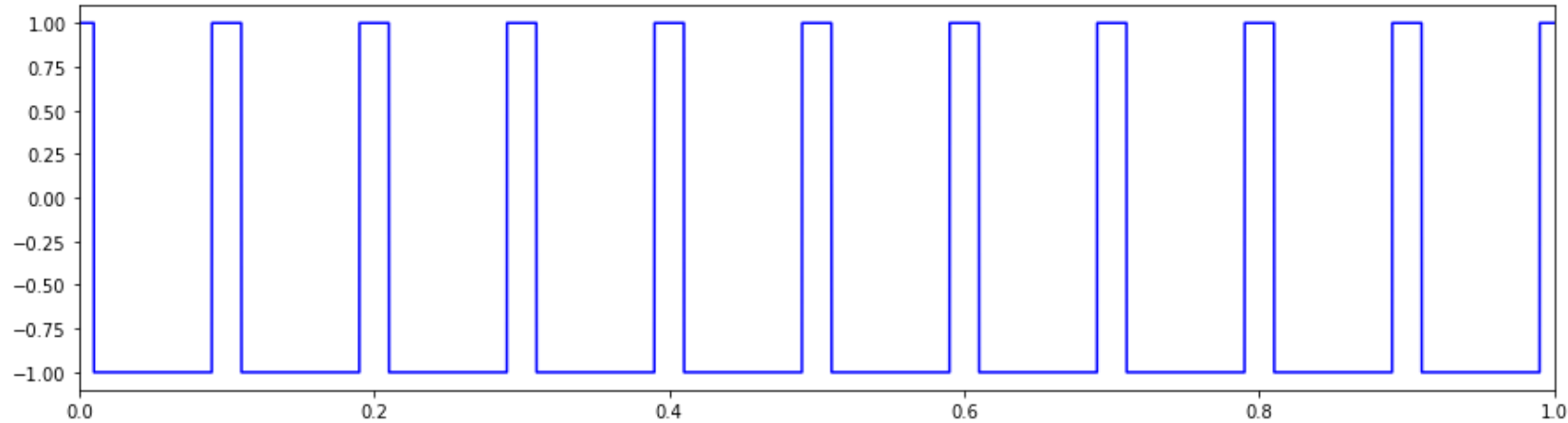
Example:

- $f_0 = 10 \text{ Hz}; T_0 = 0.1$
- Duty cycle $\frac{\tau}{T_0} = 0.2$
- $a_n = 0$ when:
 - $n = \frac{T_0}{\tau}, \frac{2T_0}{\tau}, \frac{3T_0}{\tau}, \dots$
 - $n = 5, 10, 15, \dots$
- Spectral lines at $5f_0, 10f_0, \dots$ vanish

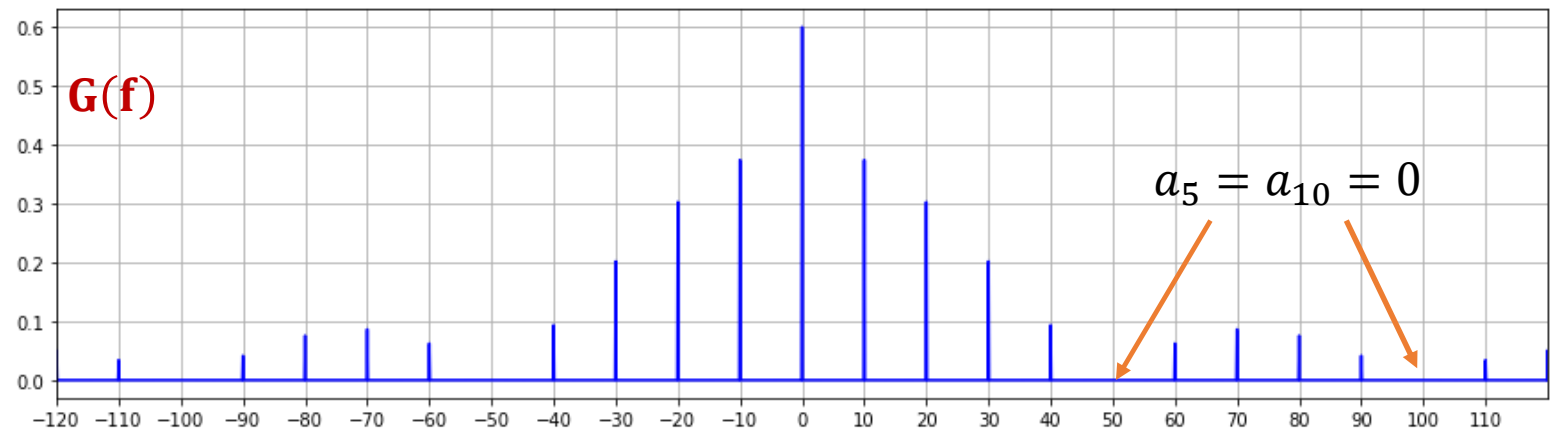
$$a_0 = A\tau/T_0$$

$$a_n = \frac{2A}{n\pi} \sin\left(\frac{n\pi\tau}{T_0}\right)$$

$g(t)$

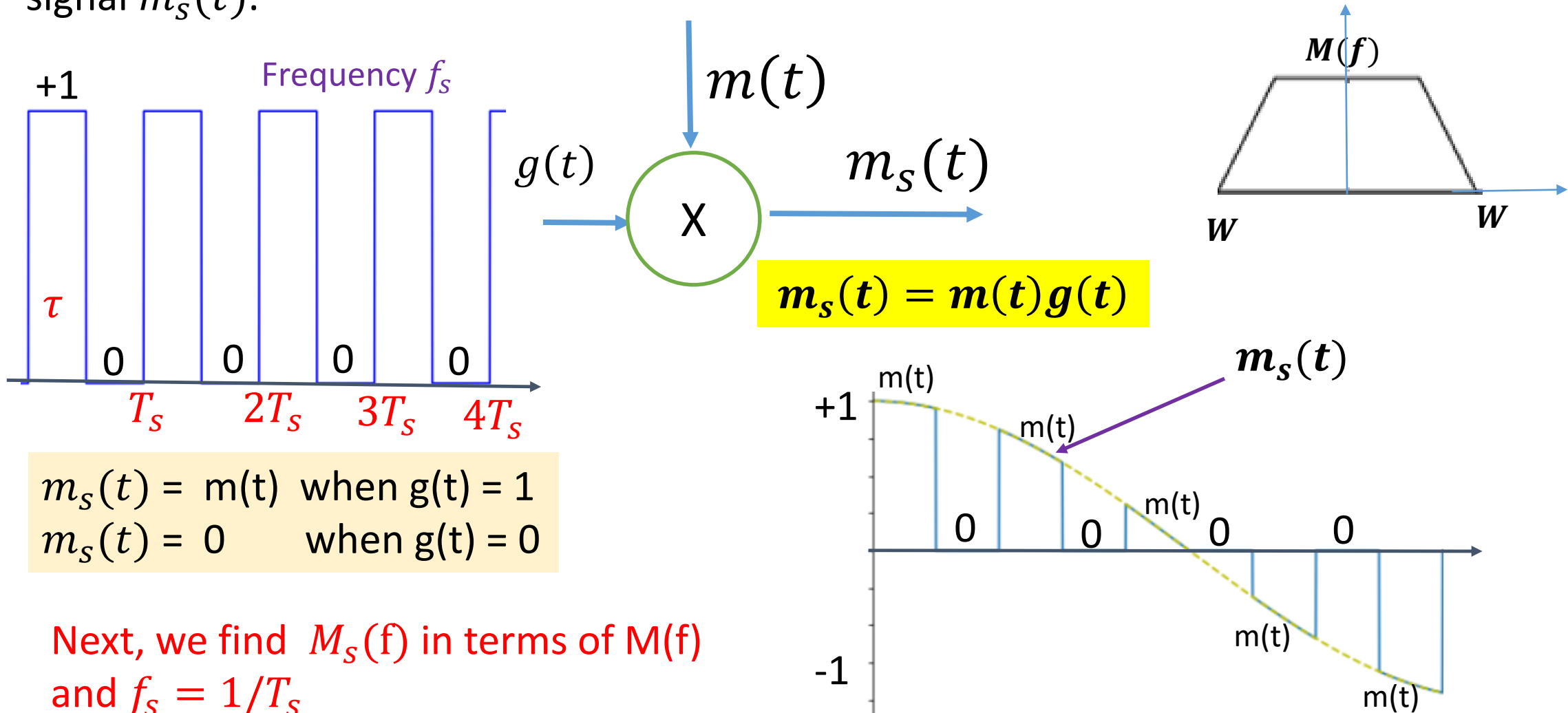


$G(f)$



Natural Sampling

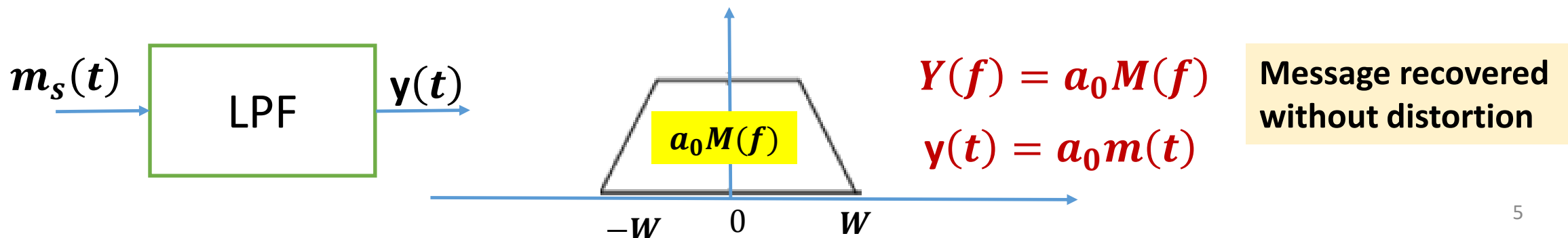
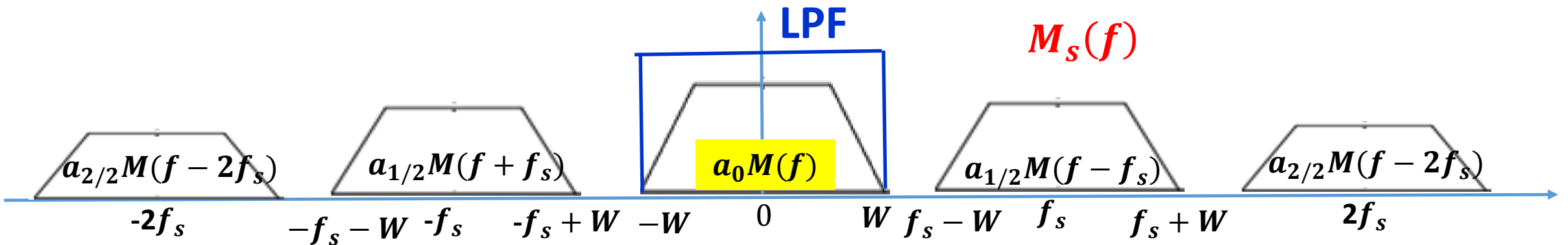
- **Natural Sampling:** The message $m(t)$, with Fourier transform $M(f)$, which is band-limited to W Hz, is multiplied by a periodic sequence of pulses with period T_s to produce the sampled signal $m_s(t)$.



Natural Sampling: $f_s > 2W$

- $m_s(t) = m(t)g(t) = m(t)\{a_0 + \sum_{n=1}^{\infty} a_n \cos 2\pi n f_s t\}$
- $m_s(t) = m(t)g(t) = a_0 m(t) + \sum_{n=1}^{\infty} a_n m(t) \cos 2\pi n f_s t$
- $M_s(f) = a_0 M(f) + \sum_{n=1}^{\infty} \frac{a_n}{2} [M(f - n f_s) + M(f + n f_s)]$
- When $f_s > 2W$, $M_s(f)$ will look like

$$f_s - W \geq W \Rightarrow f_s \geq 2W$$



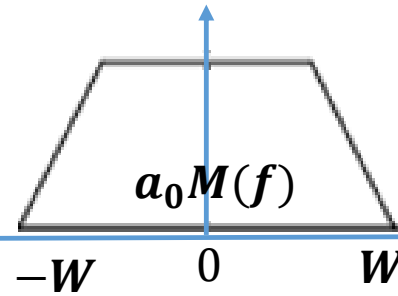
The Sampling Theorem

- A bandlimited signal with no frequency components above W Hz can be recovered uniquely from its samples taken every T_s seconds provided that $f_s \geq 2W$, where $f_s = 1/T_s$ is the sampling rate in samples/sec.
- The message $m(t)$ can be recovered from $m_s(t)$ using an ideal LPF with bandwidth W .
- The Sampling frequency $f_s = 2W$, is called the Nyquist rate. It represents the minimum rate at which a signal must be sampled in order to reconstruct it from its samples without distortion.
- When the sampling rate is less than the Nyquist rate, a distortion type of noise called **Aliasing** results.

Natural Sampling: $f_s < 2W$

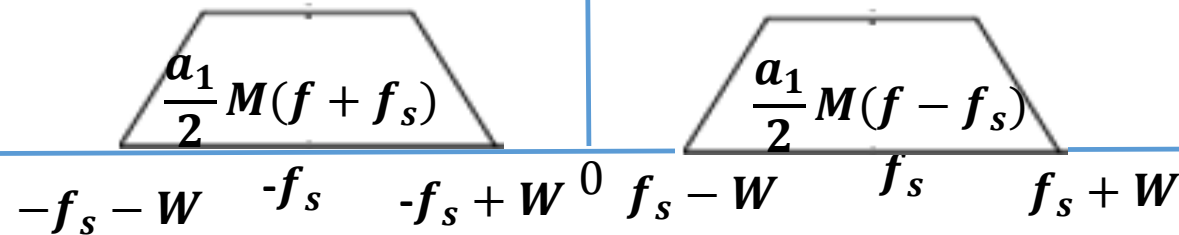
- Let $m(t)$ be the baseband signal with bandwidth W Hz. Assume that $f_s < 2W$
- Need to find $M_s(f)$ and the filtered signal.

$$M_s(f) = a_0 M(f) + \sum_{n=-\infty}^{\infty} \frac{a_n}{2} [M(f - nf_s) + M(f + nf_s)]$$



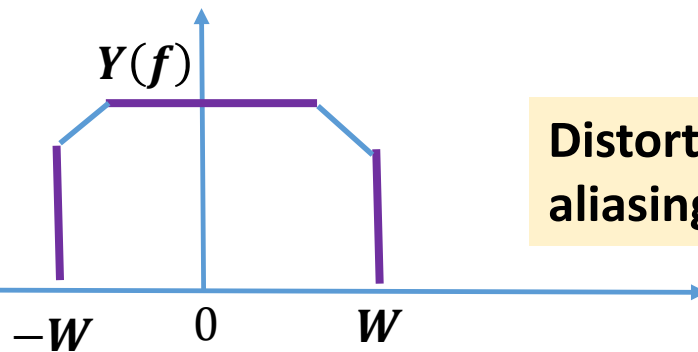
$$Y(f) = LP\{a_0 M(f) + \frac{a_1}{2} M(f - f_s) + \frac{a_1}{2} M(f + f_s)\}$$

$$f_s - W \leq W \Rightarrow f_s \leq 2W$$



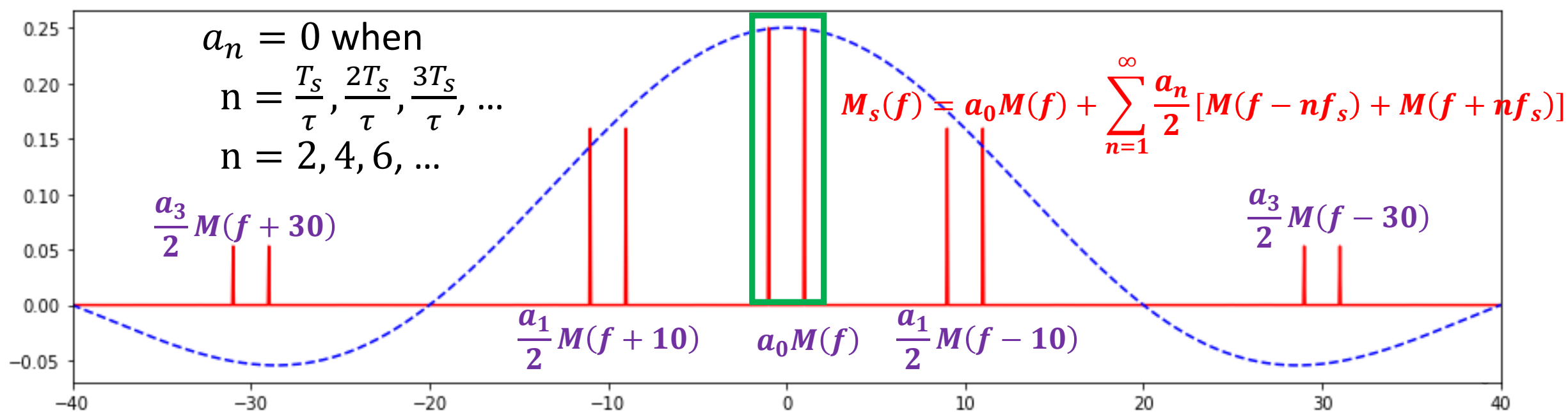
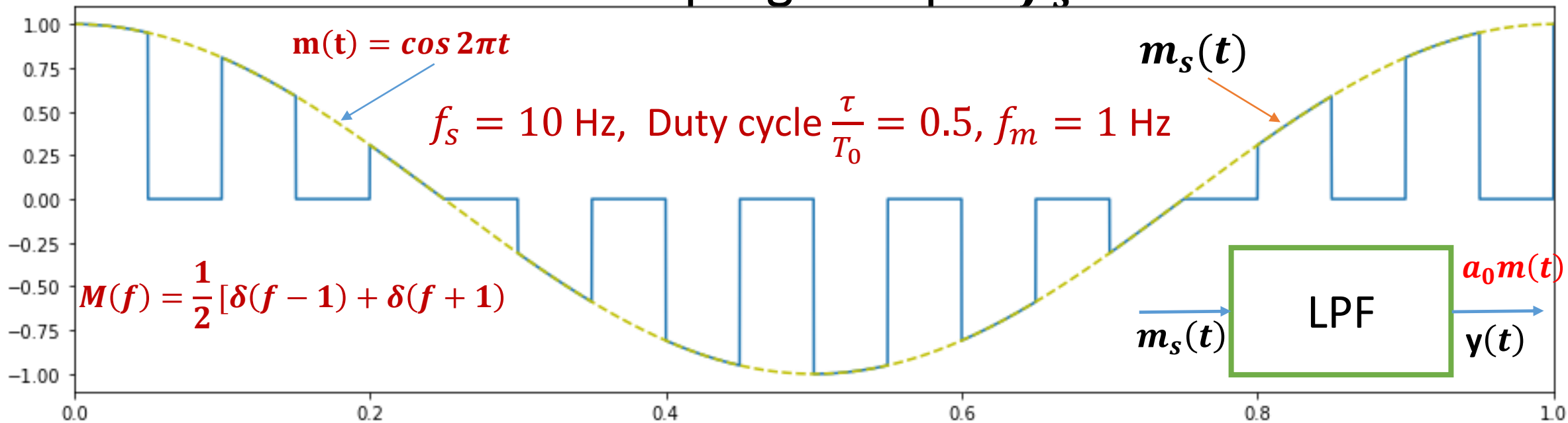
$$Y(f) \neq a_0 M(f)$$

$$y(t) \neq a_0 m(t)$$



Distortion due to aliasing is observed

Natural Sampling Example: $f_s > 2W$

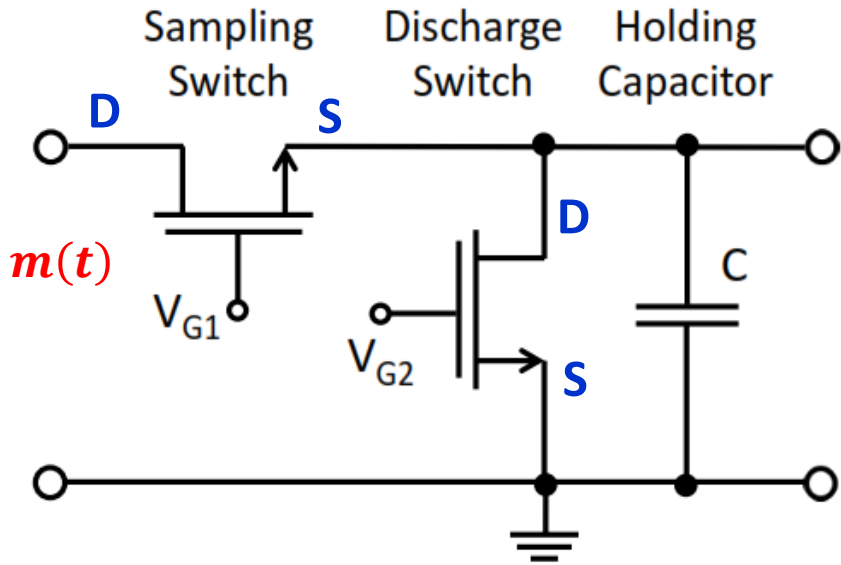


Flat-Topped Sampling

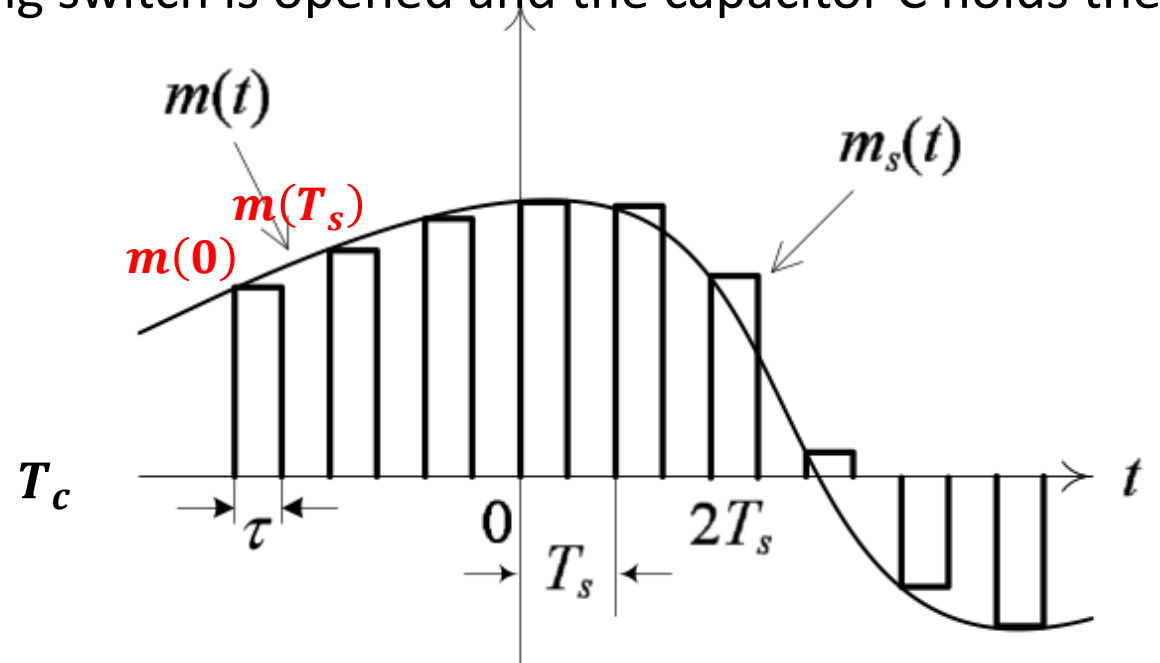
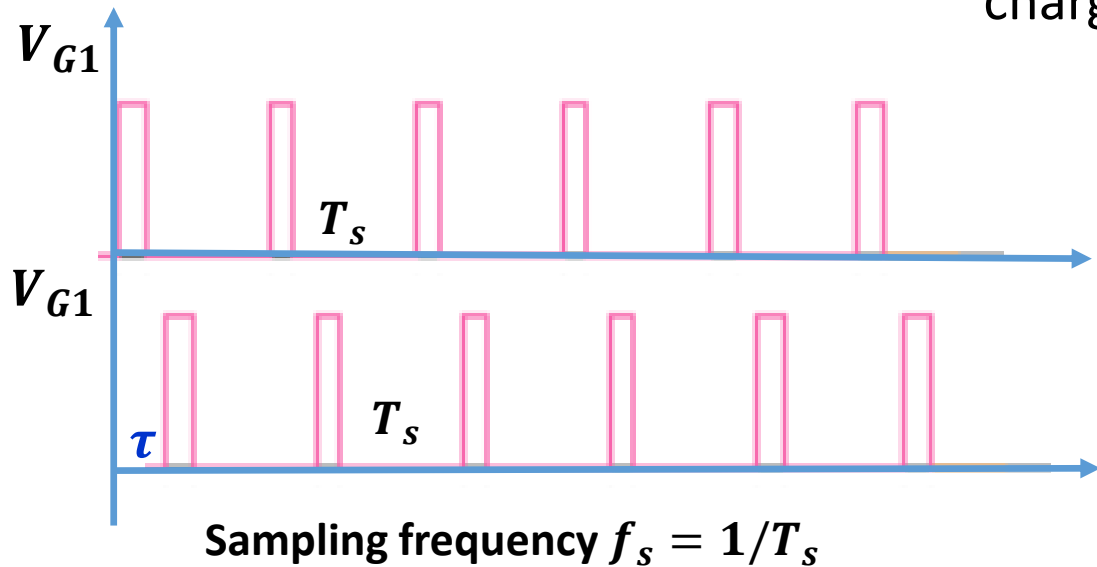
Lecture Outline

- This lecture continues the coverage of the techniques via which a continuous message signal can be sampled, as part of a PCM system
- **Sampling:** is the process by which a continuous time continuous amplitude signal is converted into a discrete time continuous amplitude signal.
- **There are three types of sampling:** Ideal sampling, natural sampling, and flat-topped sampling.
- Ideal sampling, natural sampling, the sampling theorem, and the phenomenon of aliasing were presented in the previous two lectures.
- In this lecture, we address the following topics
 - Flat-topped sampling (sample and hold).
 - Time division multiplexing (TDM)

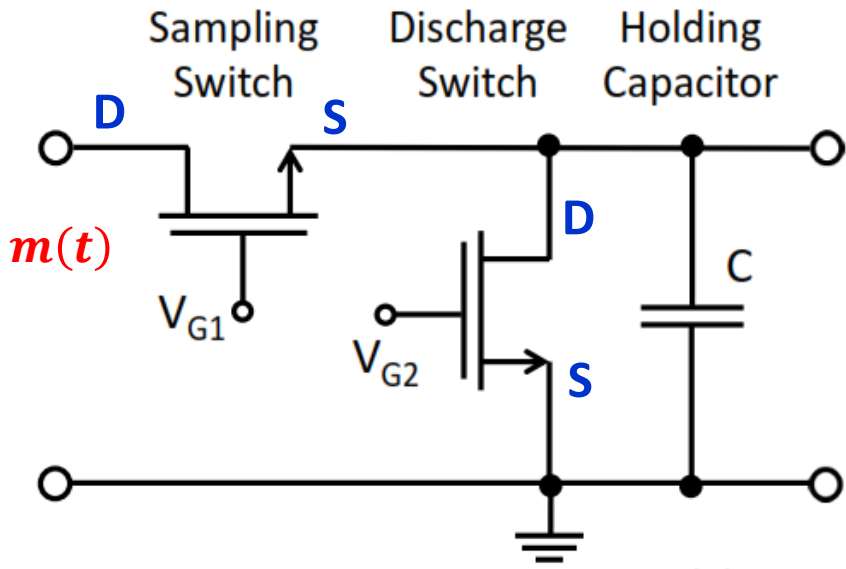
Flat-topped sampling (zero order hold sampling)



- The sample and hold circuit performs the task of sampling.
- The message $m(t)$ is bandlimited to W Hz.
- The sample and hold circuit consists of two field effect transistor (FET) switches and a capacitor.
- The sampling switch is closed for a short duration by a short pulse applied to the gate $G1$ of the transistor. During this period, the capacitor C is quickly charged up to a voltage equal to the instantaneous sample value of the incoming signal $m(t)$.
- The sampling switch is opened and the capacitor C holds the charge.

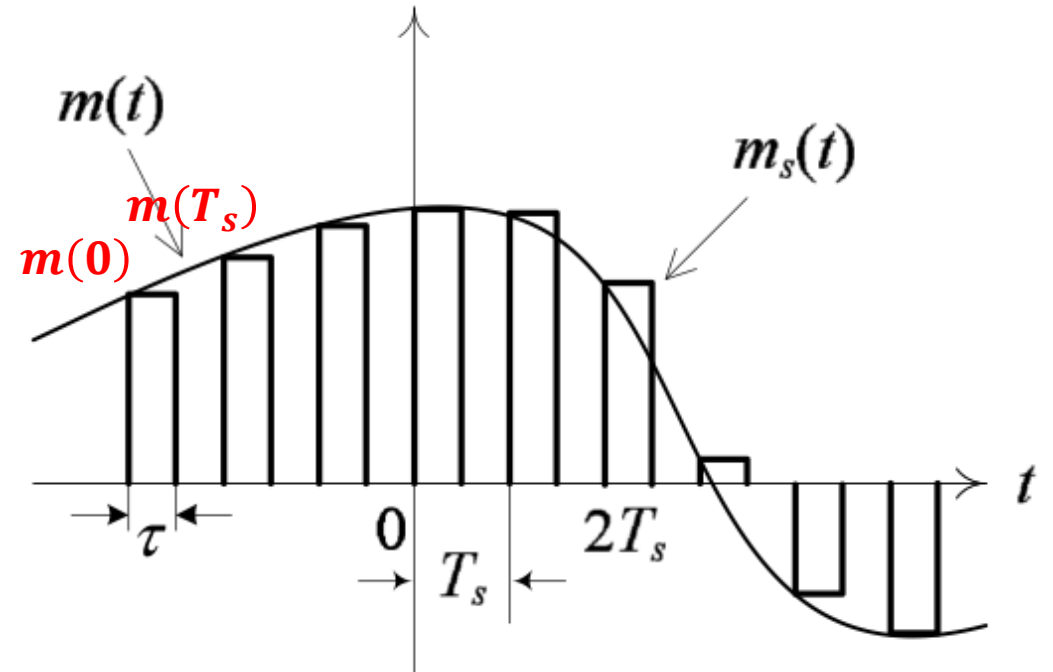
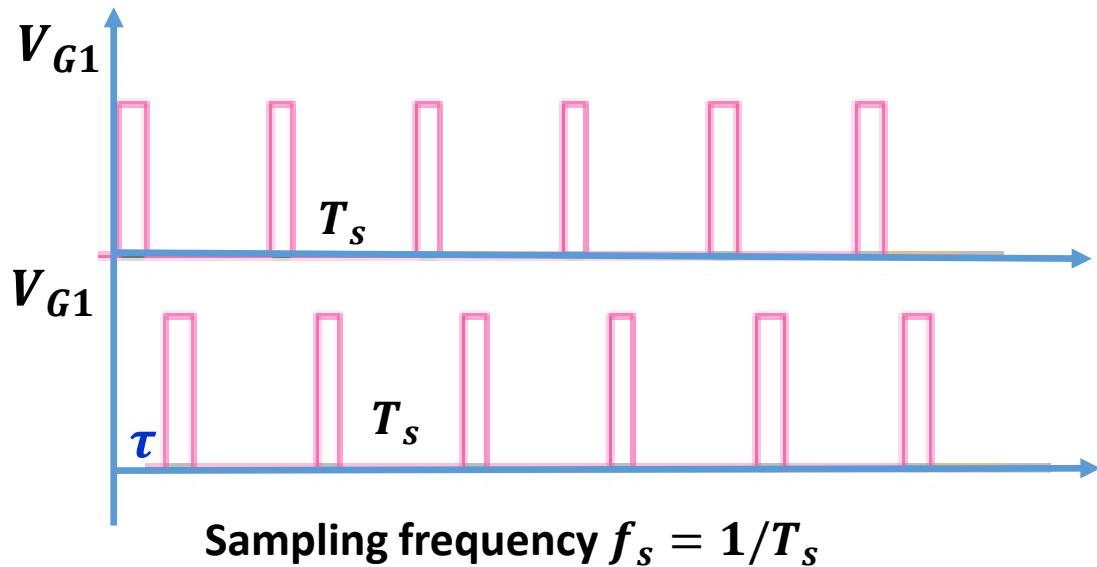


Flat-topped sampling (zero order hold sampling)



- The discharge switch is then closed by a pulse applied to its gate G2 at $t = \tau$.
- Due to this, the capacitor C is discharged to zero volts.
- The discharge switch is then opened and thus the capacitor has no voltage.
- Hence the output of the sample and hold circuit consists of a sequence of flat topped samples

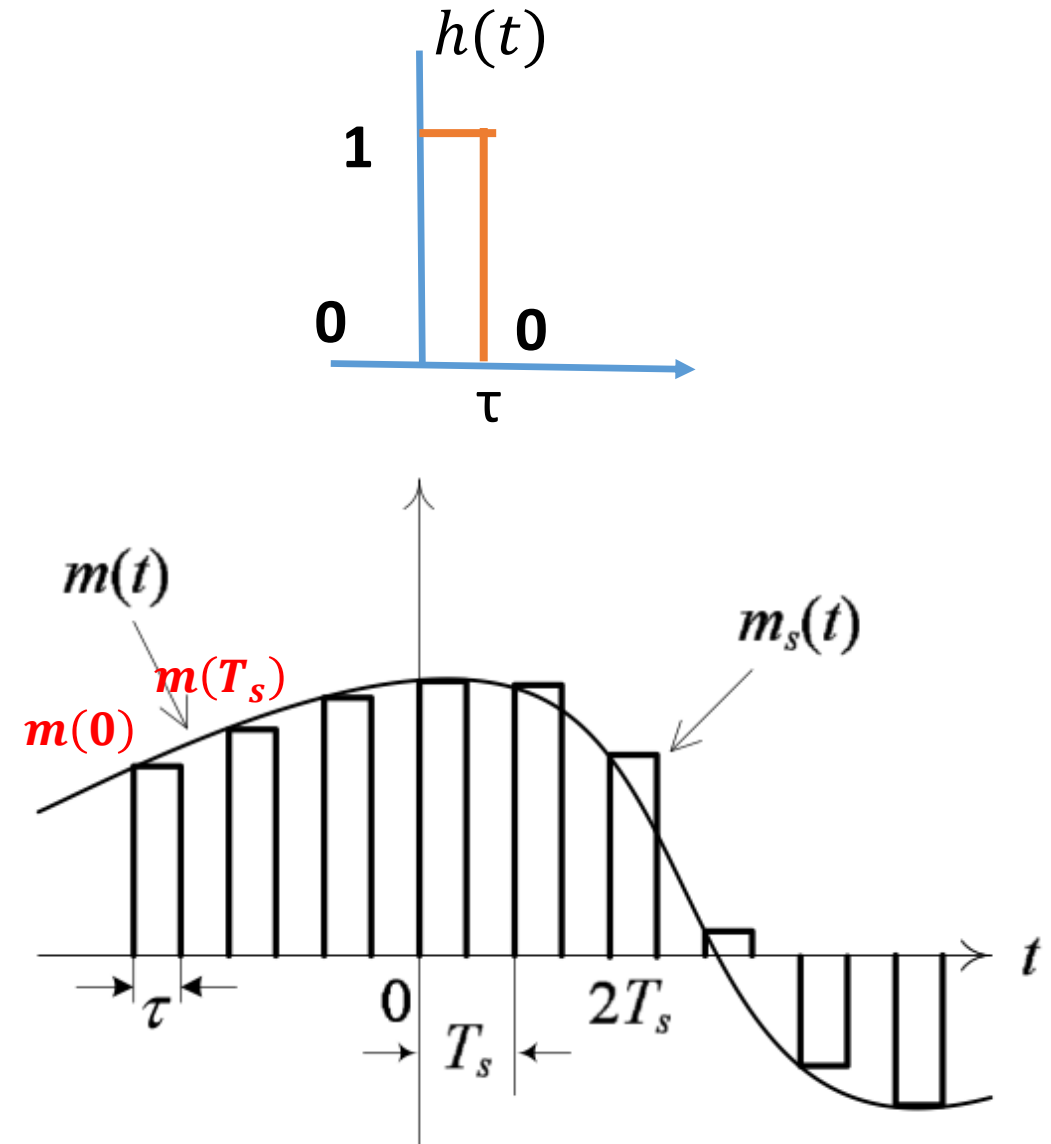
<http://technical123b.blogspot.com/2016/11/pulse-amplitude-modulation-pam.html>



Flat-topped sampling: modeling

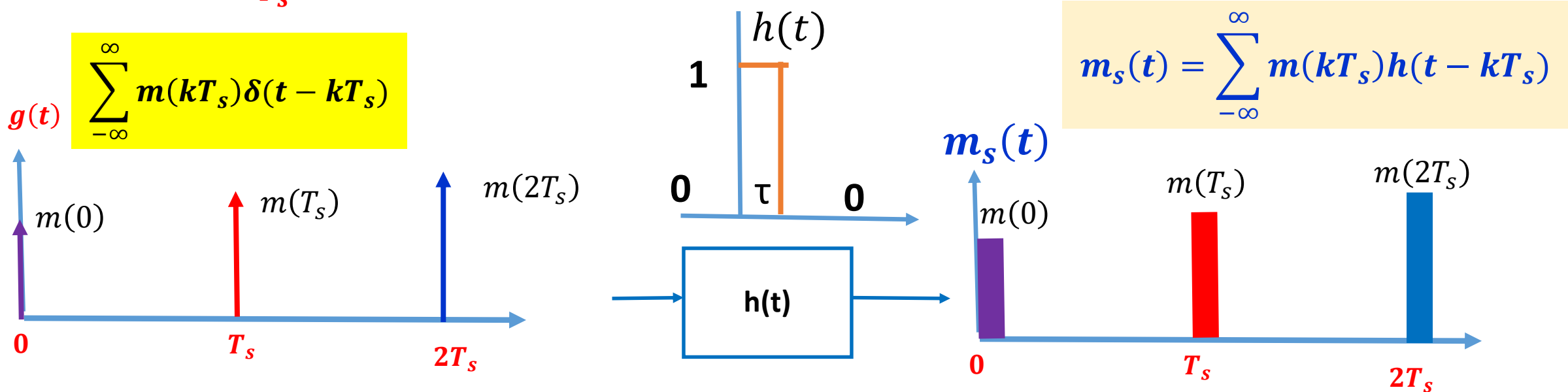
- Let $h(t)$ be a basic unit amplitude pulse defined as:
- $h(t) = \begin{cases} 1 & 0 < t < \tau, \\ 0 & \text{otherwise} \end{cases}$
- In flat-topped sampling, the sampler generates a sequence of equally spaced rectangular pulses whose amplitudes are proportional to the message signal $m(t)$ at the sampling times $m(kT_s)$.
- The sampled signal is represented as

$$m_s(t) = \sum_{-\infty}^{\infty} m(kT_s)h(t - kT_s)$$



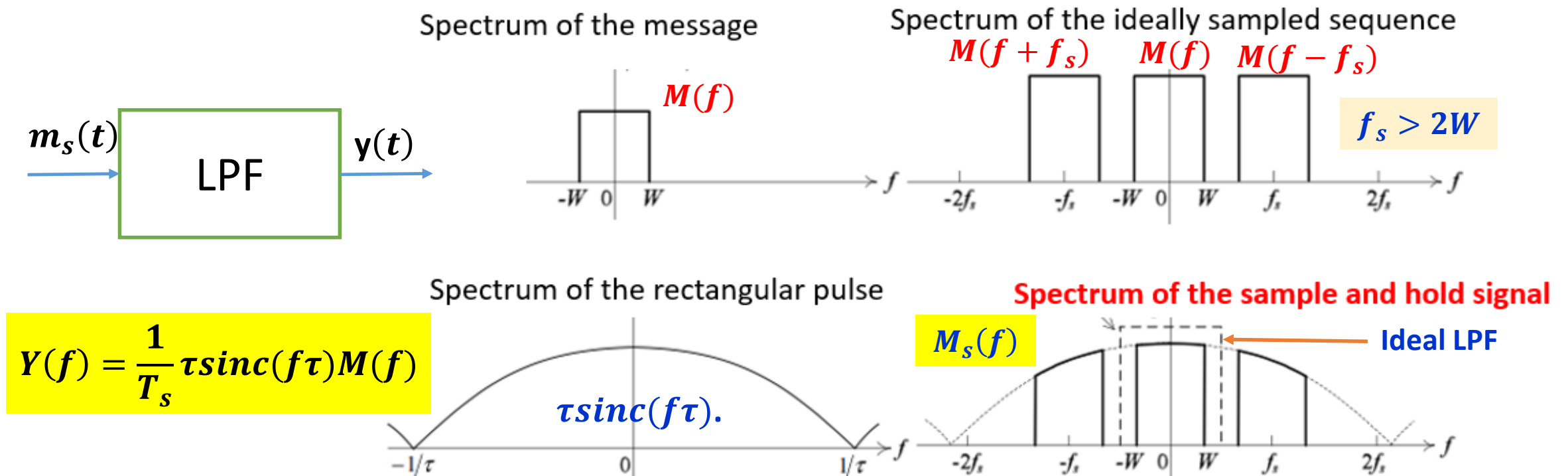
Flat-topped sampling: modeling

- The sampled signal is represented as $m_s(t) = \sum_{-\infty}^{\infty} m(kT_s)h(t - kT_s)$
- Using the identity, $\delta(t - kT_s) * h(t) = h(t - kT_s)$
- Multiplying both sides by $m(kT_s)$ we get, $m(kT_s)\delta(t - kT_s) * h(t) = m(kT_s)h(t - kT_s)$
- Therefore, $m_s(t)$ can be expressed as: $m_s(t) = h(t) * \sum_{-\infty}^{\infty} m(kT_s)\delta(t - kT_s)$
- Taking the Fourier transform, and recognizing that the second term corresponds to an ideally sampled sequence, we get
- $M_s(f) = H(f) \frac{1}{T_s} \sum_{-\infty}^{\infty} M(f - kf_s)$; where $H(f) = \tau \text{sinc}(f\tau)$.



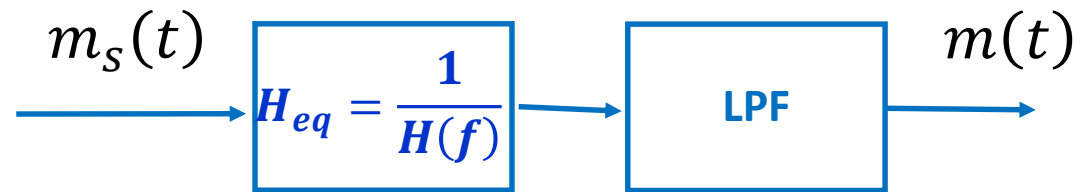
Flat-topped sampling: spectrum and message recovery

- $M_s(f) = H(f) \frac{1}{T_s} \sum_{-\infty}^{\infty} M(f - kf_s)$; $H(f) = \tau \text{sinc}(f\tau)$
- Here, we observe that the spectrum of the flat-topped sampled signal corresponds to the spectrum of the ideally sampled signal multiplied by the Fourier transform of the rectangular pulse $\tau \text{sinc}(f\tau)$.
- When $m_s(t)$ is passed through a low pass filter with bandwidth W , the output is $y(t)$



Flat-topped sampling: equalization

- The LPF filter output is $Y(f) = \frac{1}{T_s} \tau \text{sinc}(f\tau) M(f)$.
- Note that $Y(f) \neq kM(f) \Rightarrow \text{Distortion}$.
- The distortion is due to the finite width τ of the sampling pulse.
- When the message B.W $W \ll 1/\tau$, the distortion is negligible. As τ increases, the distortion becomes more pronounced.
- Even though the signal is sampled at a rate $f_s > 2W$, a distortion is observed.
- A distortion-free signal can be obtained by using an equalizing filter whose transfer function is the reciprocal of the Fourier transform of the unit pulse

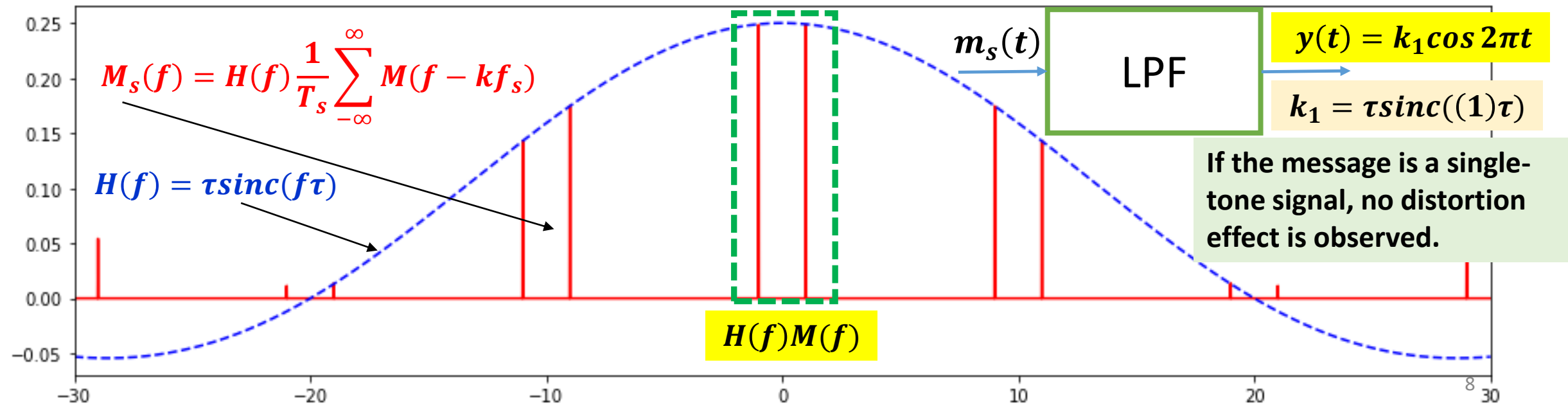
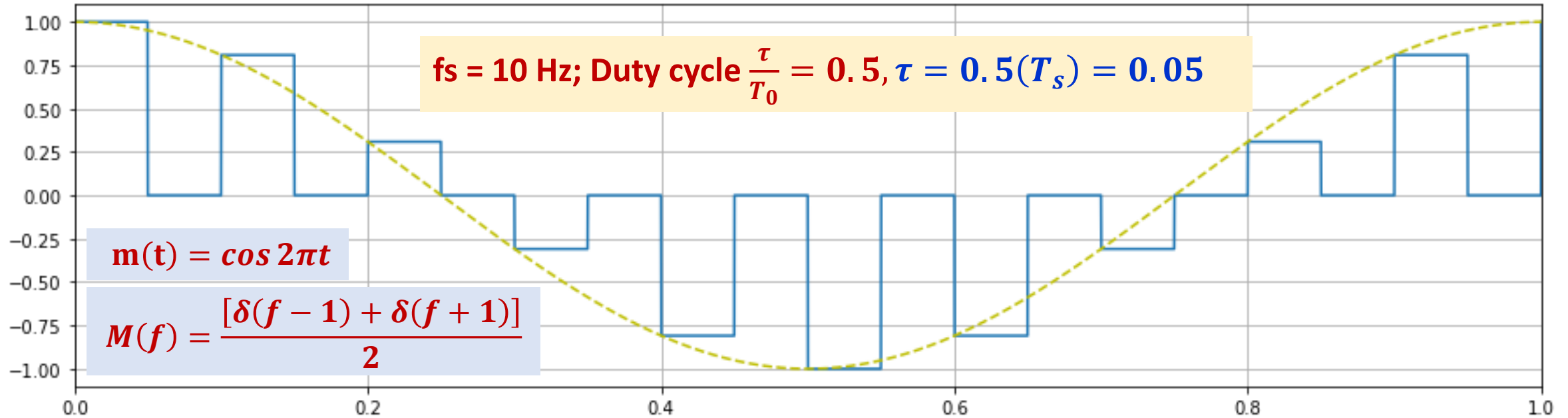


Equalization implemented at the receiver

Here,

- $f_s > 2W$
- $H(f) = A\tau \text{sinc}(f\tau)$

Flat-topped sampling: single tone example

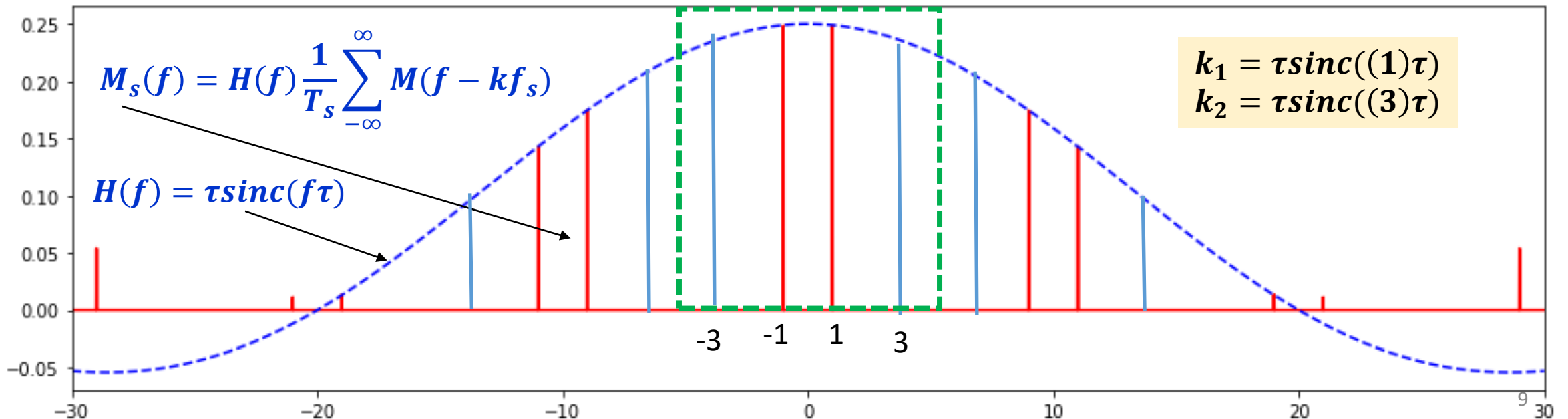


Flat-topped sampling: multi-tone example

Example: Repeat the previous example with $f_s = 10$ Hz; Duty cycle $\frac{\tau}{T_0} = 0.5$, $\tau = 0.5(T_s) = 0.05$

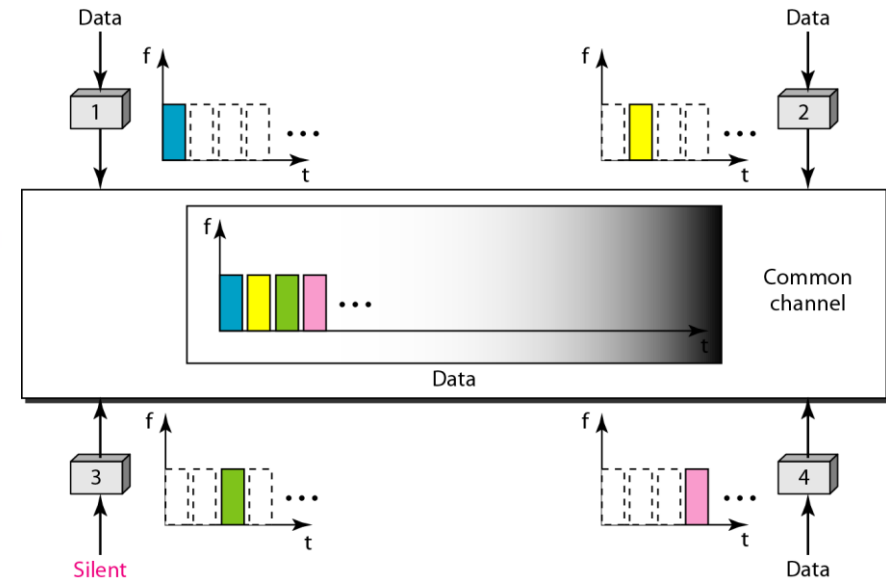
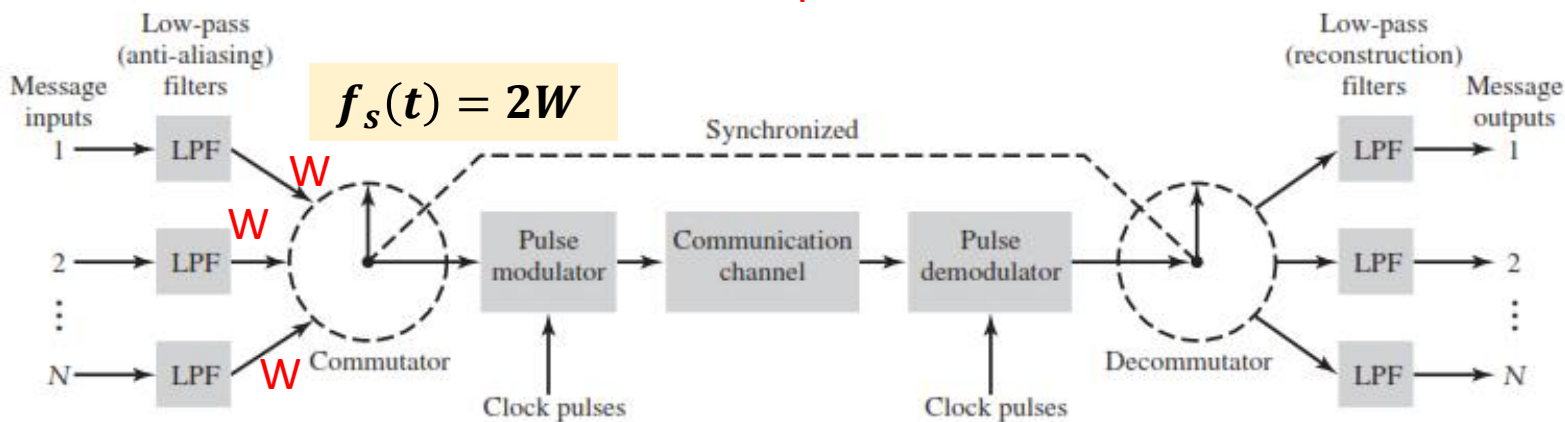
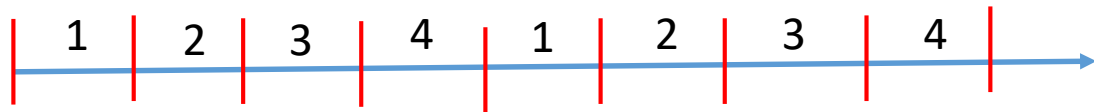
Now, let, $m(t) = \cos 2\pi t + \cos 2\pi 3t$. $M(f) = \frac{[\delta(f-1)+\delta(f+1)]}{2} + \frac{[\delta(f-3)+\delta(f+3)]}{2}$

- The spectrum of the sampled signal is as shown in the figure below.
- The output of the LPF is $y(t) = k_1 \cos 2\pi t + k_2 \cos 2\pi 3t \neq km(t)$
- Amplitude distortion is observed



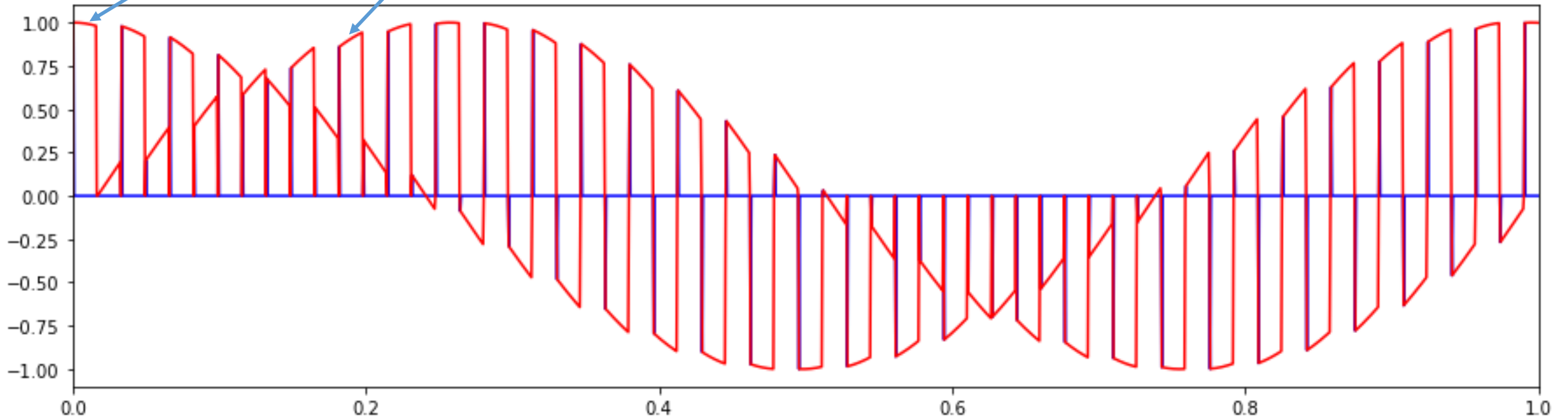
Time Division Multiplexing (TDM)

- **Time Division Multiplexing (TDM)**: A technique which allows multiple users to use the same channel by assigning each user a portion of the transmission time without interfering with other users.
- Let N be the number of sources. The time axis is divided into N slots and each slot is allocated to a source.
- Each source transmits only during its slot, avoiding the possibility of a collision.
- When a user transmits during its slot, it utilizes the entire B.W. of the channel and this B.W. will be made available to the next user during the succeeding time slot.
- The collection of the N slots is called a **cycle**.
- TDMA requires some form of synchronization.
- The number of signal samples transmitted per second should be larger than the Nyquist rate.



Time Division Multiplexing: Example

$$m_1(t) = \cos 2\pi t \quad m_2(t) = \sin 2\pi t \quad f_s(t) = 30 \quad f_{m1} = f_{m2} = 1$$



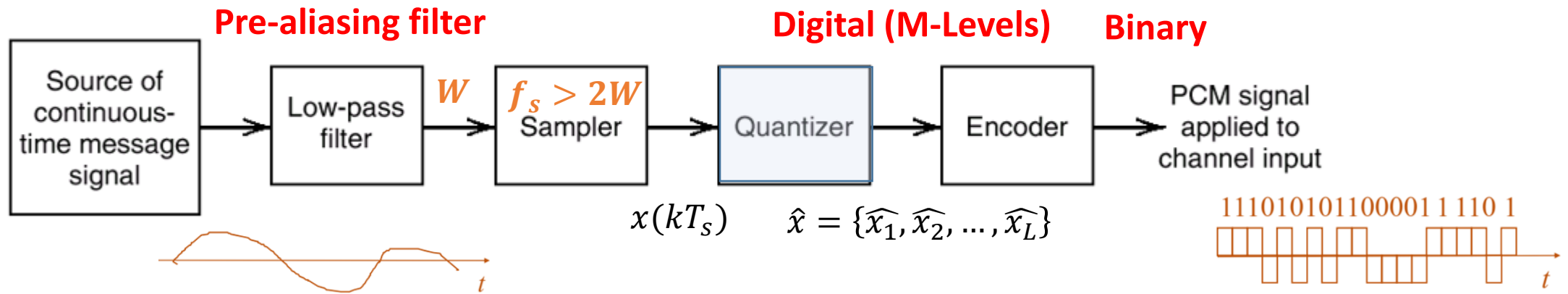
Uniform Quantization

Lecture Outline

- **Quantization** is defined as the process of converting the continuous amplitude sample $x(kT_s)$ of a message signal $x(t)$ into a discrete amplitude \hat{x} taken from a finite and countable set of L possible values.
- **In this lecture**
 - We introduce the concept of quantization.
 - Define the uniform quantizer
 - Derive the average quantization noise of the uniform quantizer.
 - Find the signal to quantization noise ratio.
 - Present a number of illustrative examples

Pulse Code Modulation: Overview

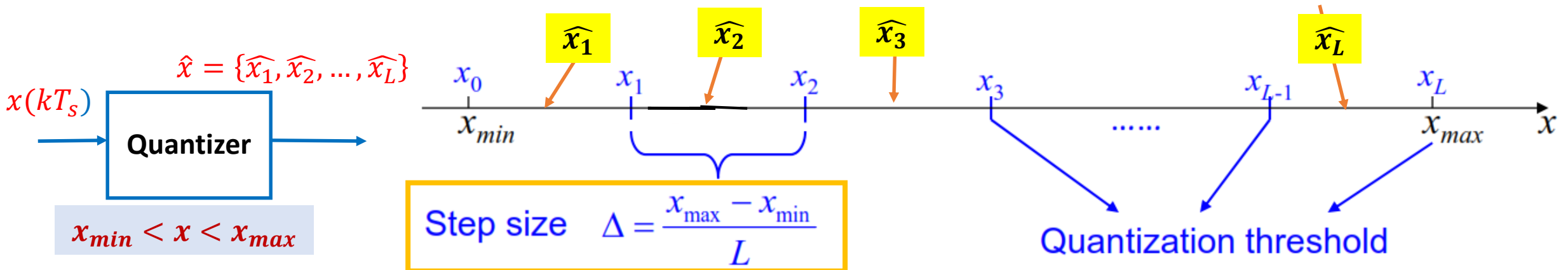
- Sources are of two types: **analog and digital**.
- An analog source can be converted into digital via sampling, quantization, and binary encoding. This process is called **pulse code modulation**



- **Sampler:** If W is the highest frequency component in a signal, then the sampling rate required to reconstruct the message from its samples should follow the Nyquist rate where $f_s > 2W$.
- **Three types of sampling were discussed in previous lectures; ideal, natural, and flat-topped.**
- The output of the sampler is a continuous amplitude discrete time signal.
- **Quantizer:** Converts the continuous amplitude samples $x(kT_s)$ into **discrete** level samples $\hat{x}(kT_s)$ taken from a finite set of L possible values $\hat{x} = \{\hat{x}_1, \hat{x}_2, \dots, \hat{x}_L\}$.
- **Binary Encoder:** Each quantized level is represented by $r = \log_2 L$ binary digits

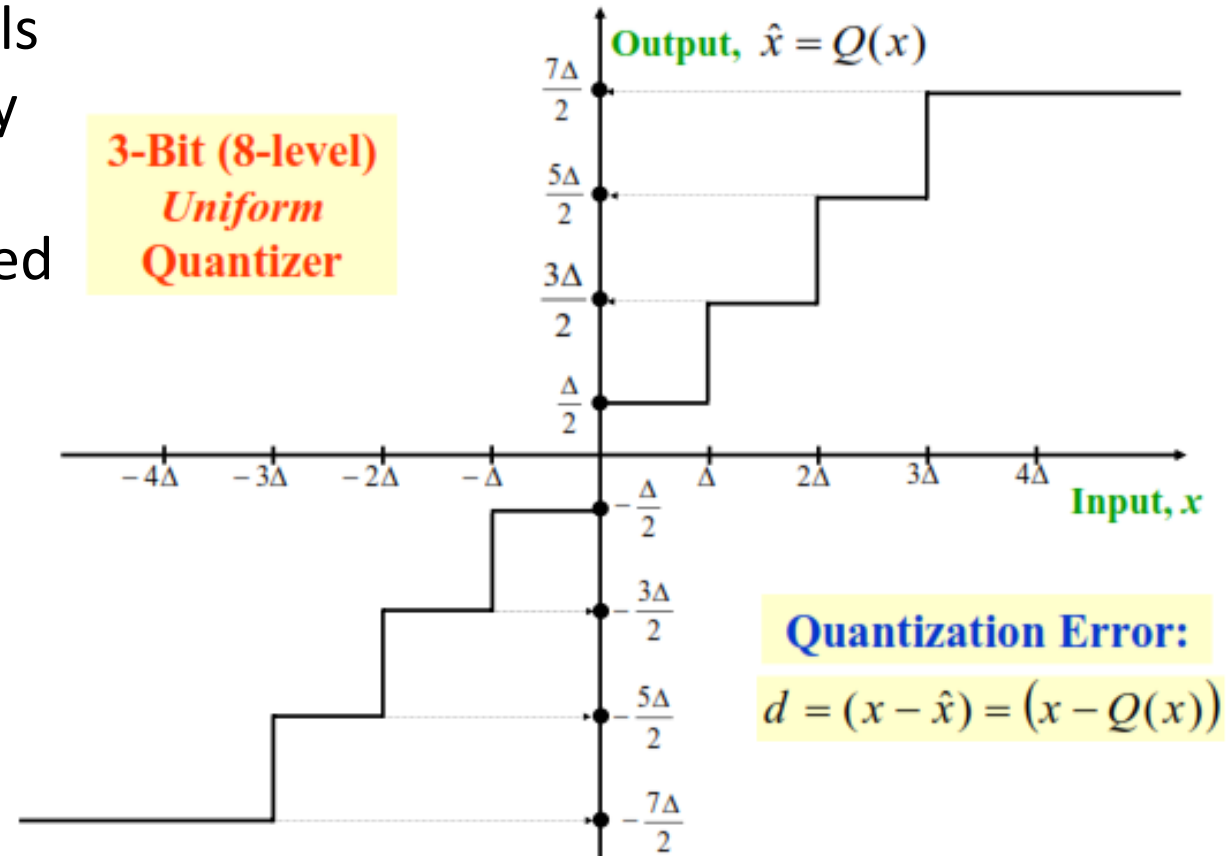
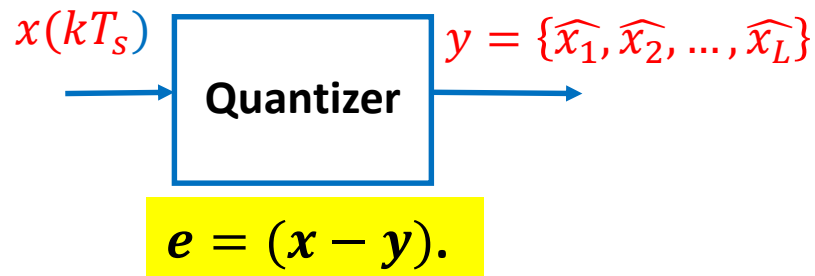
Quantization: Basic Definitions

- **Quantization**: is defined as the process of converting the continuous amplitude sample $x(kT_s)$ of a message signal into a discrete amplitude $\hat{x}(kT_s)$ taken from a finite and countable set of L possible values $\hat{x} = \{\hat{x}_1, \hat{x}_2, \dots, \hat{x}_L\}$.
- The **dynamic range** of the quantizer is the range of values for which the quantizer is designed, $x_{min} < x < x_{max}$
- This range is partitioned into L intervals such that if $x(kT_s) \in R_i$, the quantizer output will be a level $\hat{x}_i = \{\hat{x}_1, \hat{x}_2, \dots, \hat{x}_L\}$
- The quantizer output is called a **representation or reconstruction level**
- The boundary points separating adjacent regions are called **decision or threshold levels**.
- The spacing between representation levels is called the **step size (Δ)**



The Uniform Quantizer: Input-Output Characteristic

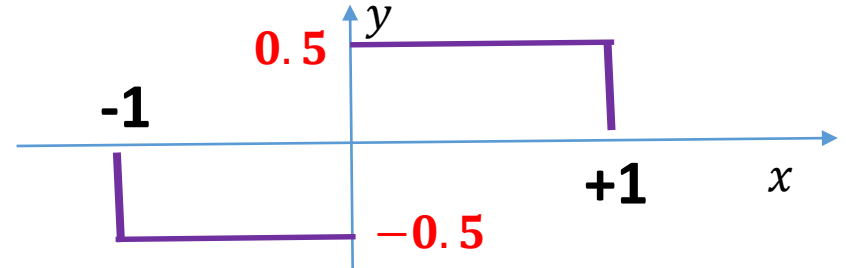
- A quantizer is called **uniform** when the L regions are of equal length Δ and the spacing between representation levels is uniform and equals to Δ .
- The input-output characteristic of a uniform quantizer (midrise type) is shown below for $L=8$.
- If the dynamic range of the quantizer varies between $x_{min} < x < x_{max}$, then $\Delta = \frac{x_{max} - x_{min}}{L}$
- When the spacing between the adjacent levels is made small, $\hat{x}(kT_s)$ can be made practically indistinguishable from $x(kT_s)$.
- There is always a loss of information associated with the quantization process. Therefore, it is not possible to completely recover the sampled signal from the quantized signal.



Example: the one-bit quantizer

- Example:** The signal $x(t) = \cos(2\pi t)$ is uniformly sampled at a rate of 20 samples per second. The samples are applied to a sign detector, whose input-output characteristic is defined as:

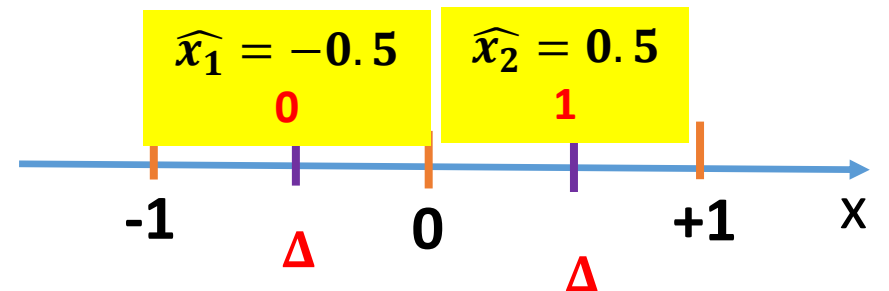
$$y(t) = \begin{cases} 0.5, & 0 < x < 1 \\ -0.5, & -1 < x < 0 \end{cases}$$



- The next figures depict the input samples to the sign detector, the quantized output, and the quantization error defined as $e = (x - y)$.

- Here, $\Delta = \frac{1 - (-1)}{2} = 1$;

t=	0	0.0500	0.1000	0.1500	0.2000	0.2500
x(t)=	1.0000	0.9511	0.8090	0.5878	0.3090	0.0000
y =	0.5	0.5	0.5	0.5	0.5	0.5
e =	0.5	0.4511	0.3090	0.0878	-0.191	-0.5
Note that	$-\frac{\Delta}{2} < e < \frac{\Delta}{2} \Rightarrow -0.5 < e < 0.5$					

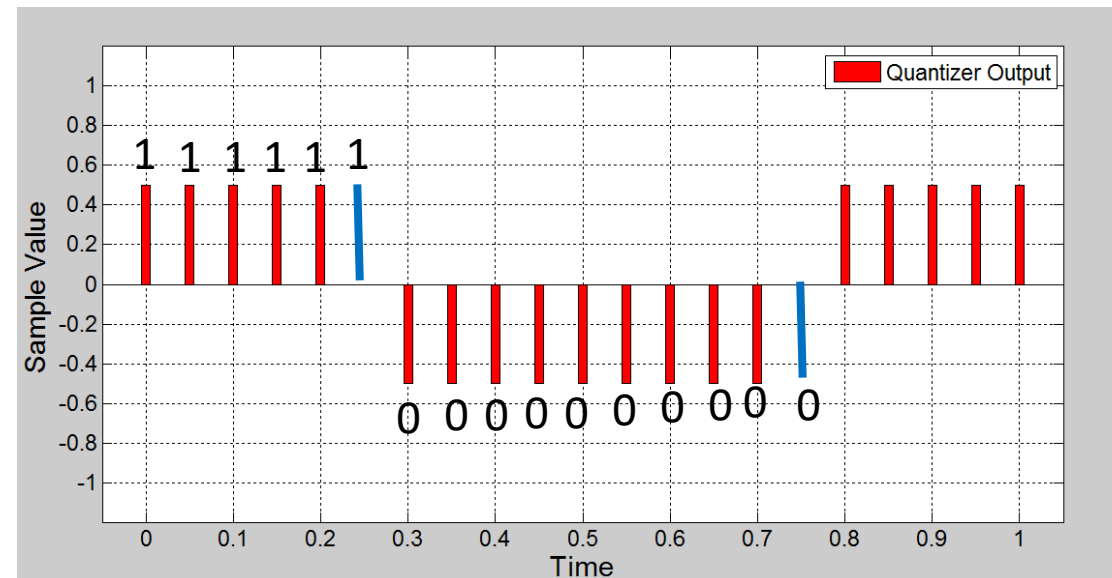
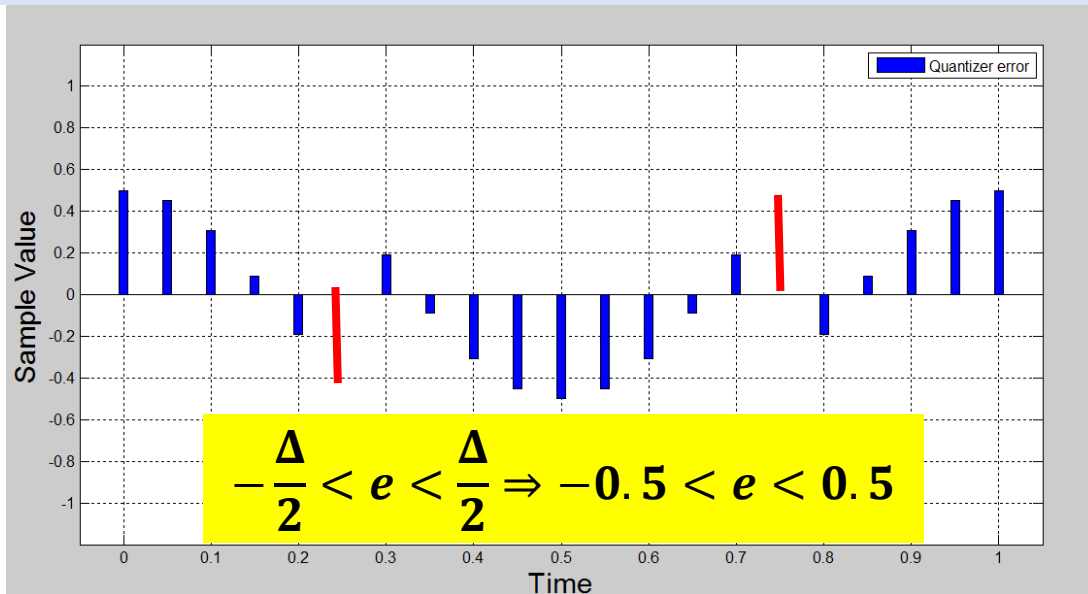
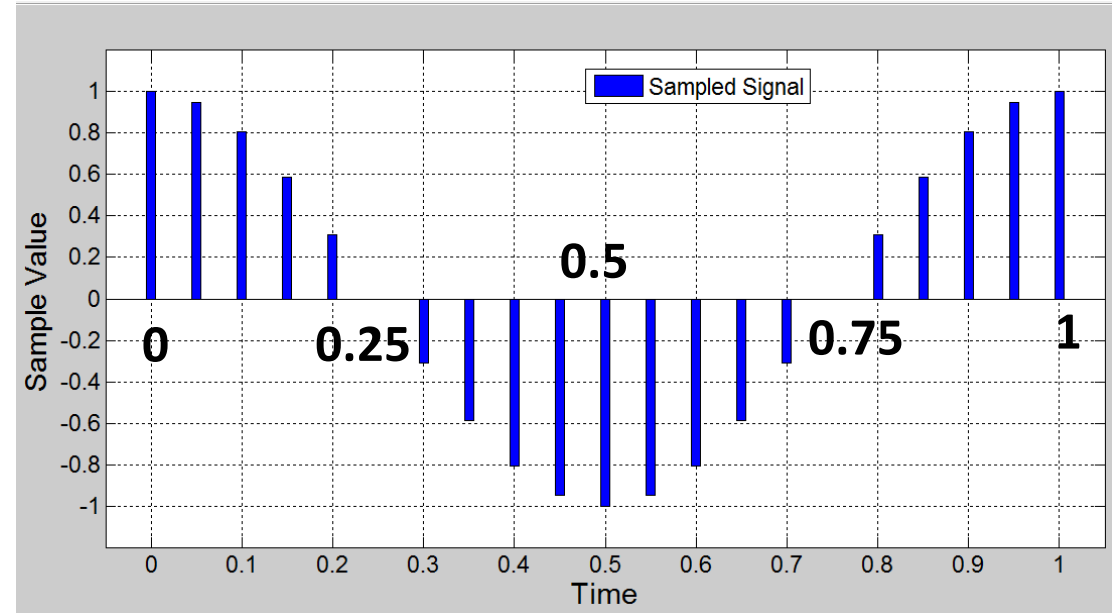


Quantization: the one-bit quantizer

- **Example:** The signal $x(t) = \cos(2\pi t)$ is uniformly sampled at a rate of 20 samples per second.
- The samples are applied to a sign detector.
- Binary digits are assigned to the quantizer output.

$t =$	0	0.0500	0.1000	0.1500	0.2000	0.2500
$x(t) =$	1.0000	0.9511	0.8090	0.5878	0.3090	0.0000
$y =$	0.5	0.5	0.5	0.5	0.5	0.5
$e =$	0.5	0.4511	0.3090	0.0878	-0.191	-0.5

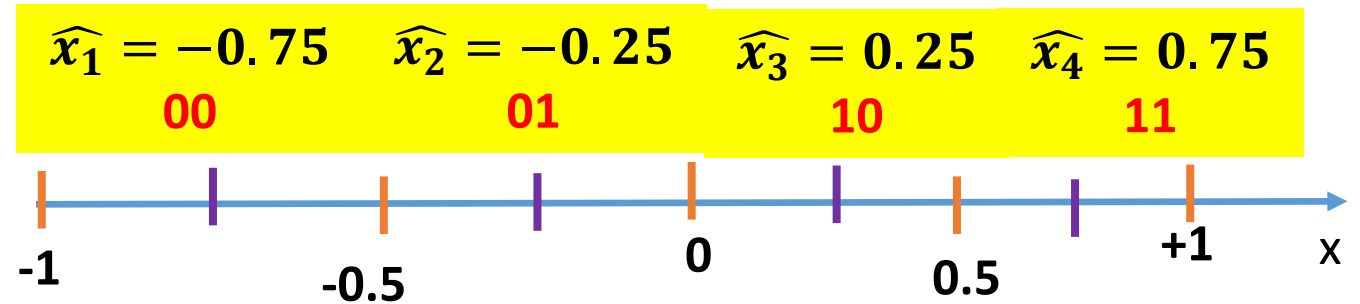
Note that $-\frac{\Delta}{2} < e < \frac{\Delta}{2} \Rightarrow -0.5 < e < 0.5$



Quantization: the two-bit quantizer

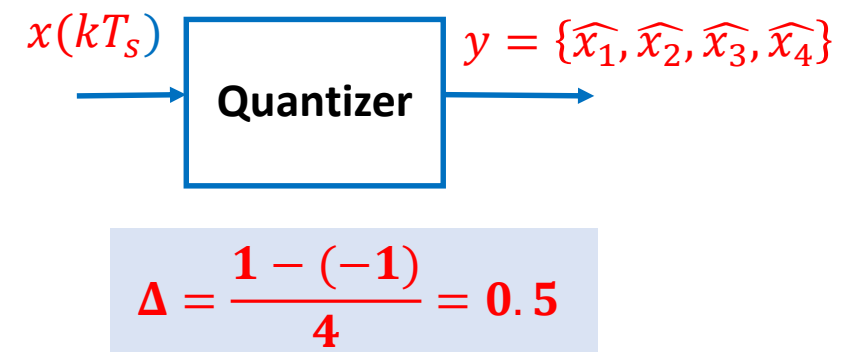
- Example:** The signal $x(t) = \cos(2\pi t)$ is sampled uniformly at a rate of 20 samples per second. The samples are applied to a four-level uniform quantizer with input-output characteristic

$$y(t) = \begin{cases} 0.75, & 0.5 < x < 1 \\ 0.25, & 0 < x < 0.5 \\ -0.25, & -0.5 < x < 0 \\ -0.75, & -1 < x < -0.5 \end{cases}$$



- The next figures depict the quantizer operation, the input samples to the sign detector, the quantized output, and the quantization error defined as $e = (x - y)$.

$t =$	0	0.0500	0.1000	0.1500	0.2000	0.2500
$x(t) =$	1.0000	0.9511	0.8090	0.5878	0.3090	0.0000
$y =$	0.75	0.75	0.75	0.75	0.25	0.25
$e =$	0.25	0.2011	0.059	-0.1622	0.059	-0.25
Note that	$-\frac{\Delta}{2} < e < \frac{\Delta}{2} \Rightarrow -0.25 < e < 0.25$					

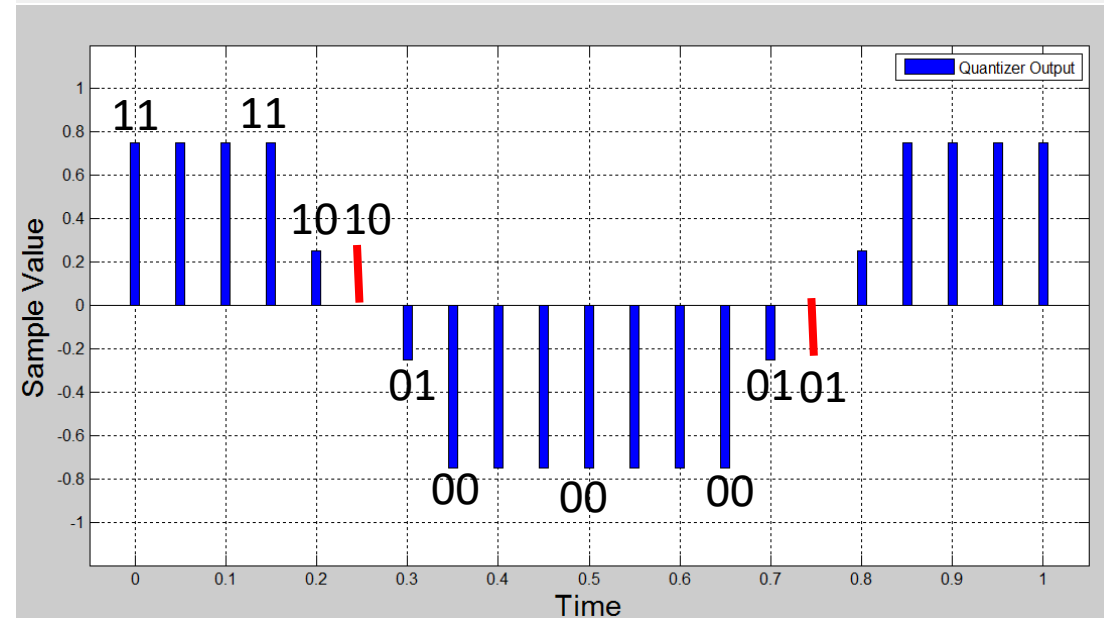
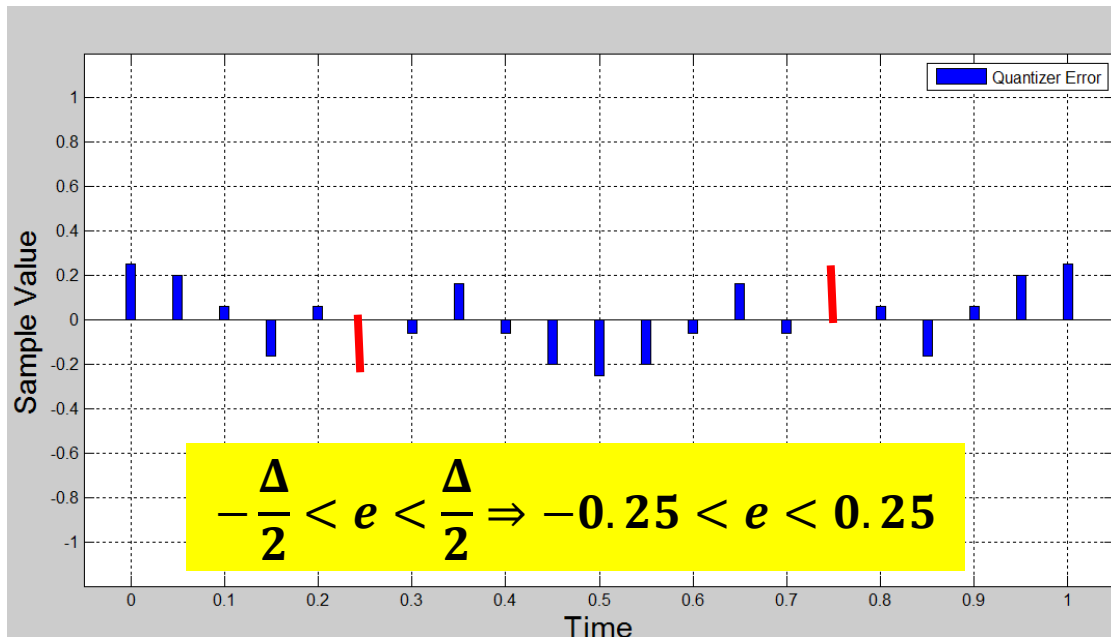
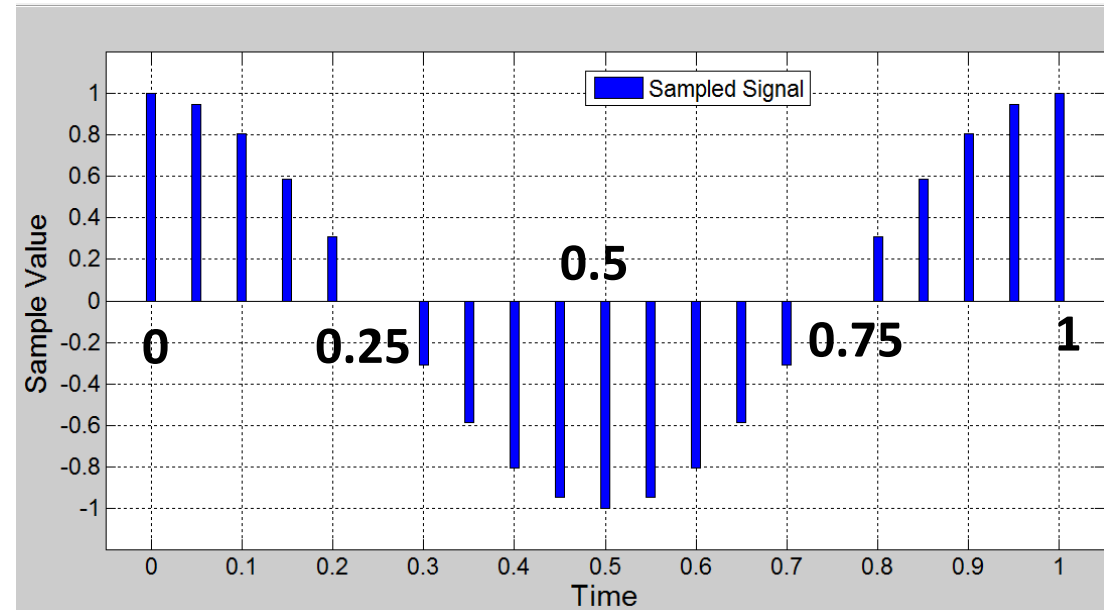


Quantization: the two-bit quantizer

- Example:** The signal $x(t) = \cos(2\pi t)$ is sampled uniformly at a rate of 20 samples per second. The samples are applied to a four-level uniform quantizer
- Binary digits are assigned to the quantizer output

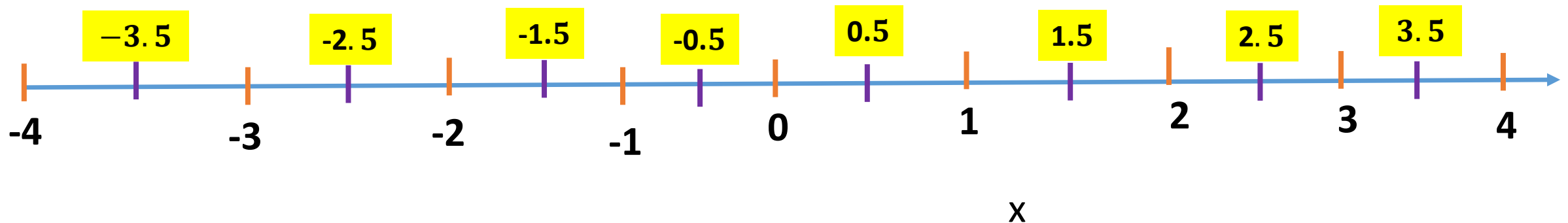
$t =$	0	0.0500	0.1000	0.1500	0.2000	0.2500
$x(t) =$	1.0000	0.9511	0.8090	0.5878	0.3090	0.0000
$y =$	0.75	0.75	0.75	0.75	0.25	0.25
$e =$	0.25	0.2011	0.059	-0.1622	0.059	-0.25

Note that $-\frac{\Delta}{2} < e < \frac{\Delta}{2} \Rightarrow -0.25 < e < 0.25$



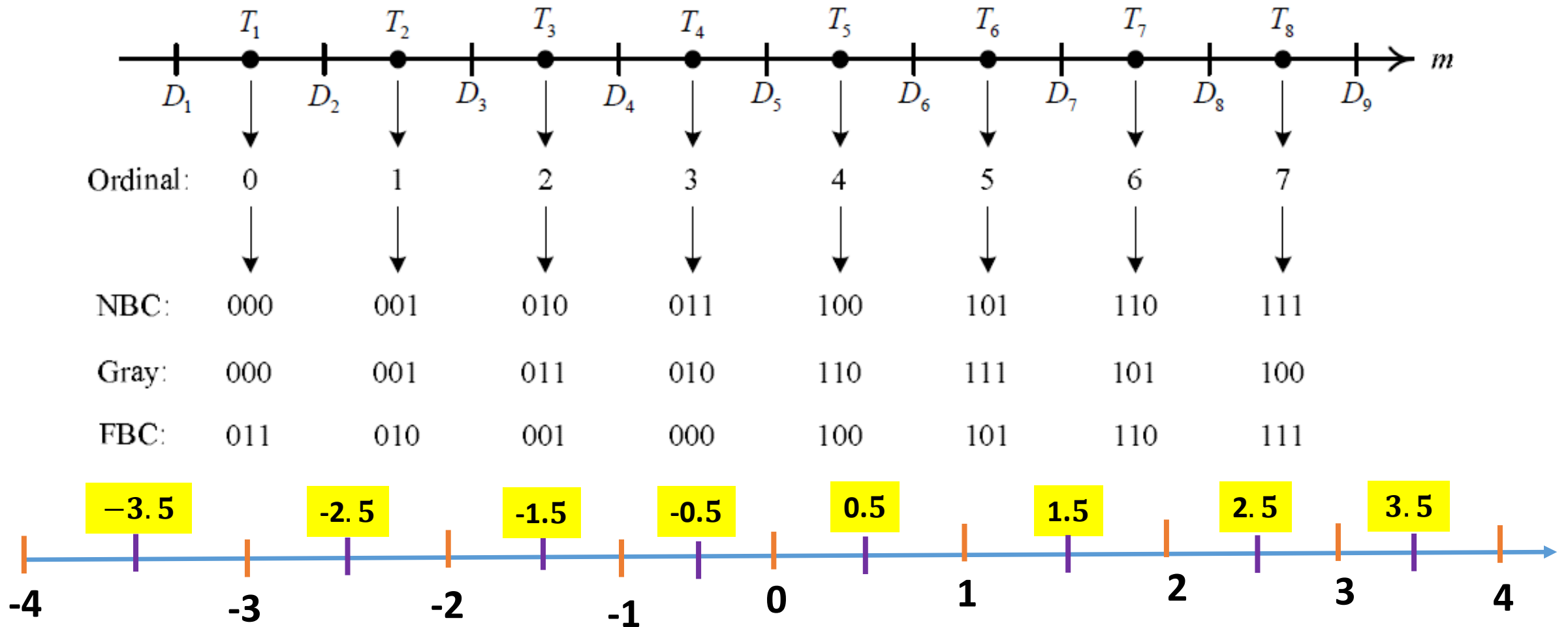
Example: Quantizer Design

- Design an 8-level uniform quantizer with a dynamic range of (-4, +4) V. Here, you need to specify the thresholds and the representation values.
- How many binary digits are needed to represent the samples? **(3 bits)**
- Find the representation value and the quantization error when a 1.64 V sample is applied to the quantizer.
- **Solution:** $\Delta = \frac{4 - (-4)}{8} = 1$
- Since $L = 8$, then $n=3$. $L = 2^n; n = \log_2(L)$
- When $x = 1.64$, $y = 1.5$, and the error is: $e = (x - \hat{x}) = (1.64 - 1.5) = 0.14$.



Example: continued

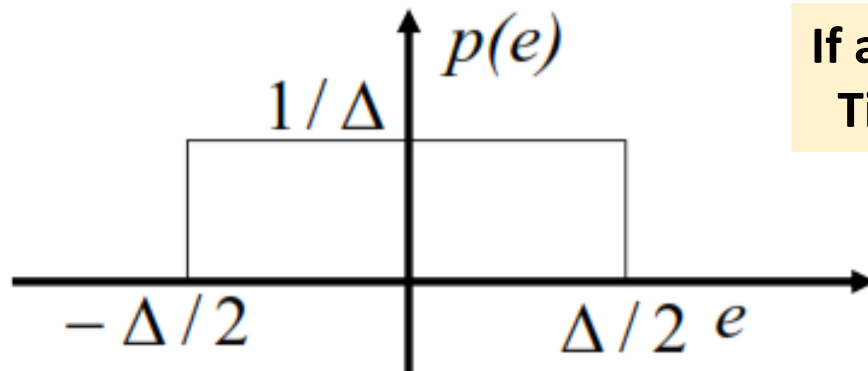
- Find the binary representation corresponding to the sample -2.1 V if natural binary encoding is employed (**-2.5; 001**)
- Are there other binary encoding formats?



Quantization Noise

- The **quantization error** per sample is the difference between the input and output of the quantizer, i.e., $e = (x - \hat{x})$
- The time average of the mean squared error $\frac{1}{T} \int (x - \hat{x})^2 dt$; **(Time Average)**
- The maximum error (also referred to as the resolution) = $|\frac{\Delta}{2}|$
- When Δ is small, the error, e , is assumed to be a uniform random variable over the interval $-\Delta/2 < e < \Delta/2$.
- The average quantization error (distortion) over all samples of the signal is
- $D = E(x - \hat{x})^2 = E(e^2) \Rightarrow D = \frac{1}{\Delta} \int_{-\Delta/2}^{+\Delta/2} (e)^2 de \Rightarrow \mathbf{D = \Delta^2/12}$ **(Statistical Average)**
- **Remark:** Note that D depends on the design of the quantizer and not on the signal applied to it, as we will see in the next two examples.

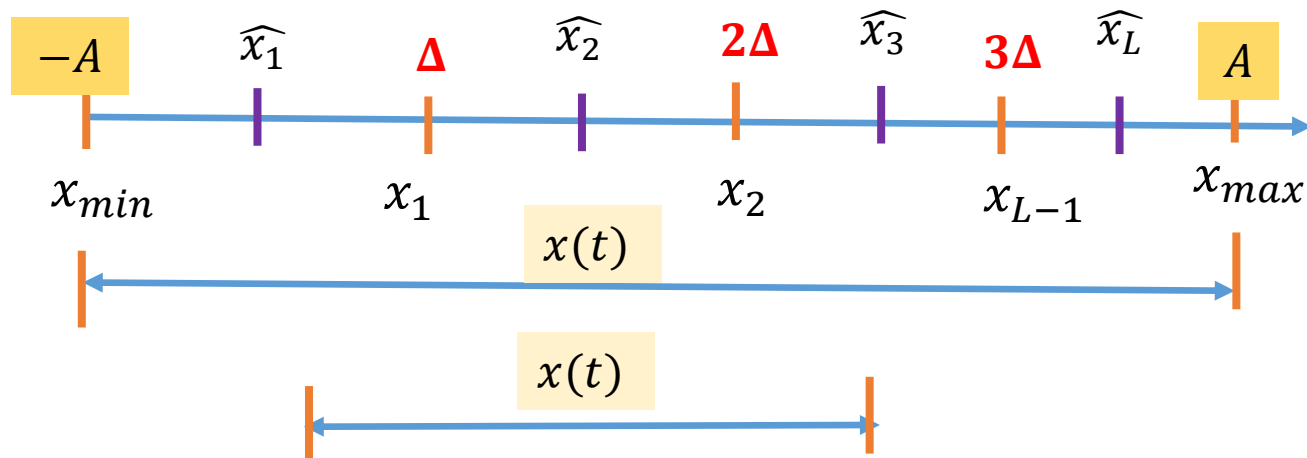
$$\left[-\frac{\Delta}{2}, \frac{\Delta}{2} \right]$$



If a process is assumed ergodic, then
Time Average = Statistical Average

SQNR: Signal Matches Dynamic Range of a Uniform Quantizer

- **Example:** Let the sinusoidal signal $x(t) = A\cos(2\pi f_0 t)$ be applied to a uniform quantizer with a dynamic range $(-A, A)$. We need to find the *SQNR*.
- **Solution:** The average power, P_x , in $x(t)$ is: $P_x = A^2/2$
- The signal power to quantization noise ratio: $SQNR = \frac{A^2/2}{\Delta^2/12}$
- Here, $\Delta = 2A/L$. If $L = 2^n$, then the *SQNR* become: $SQNR = \frac{3}{2}L^2 = \frac{3}{2}2^{2n}$
- In dB, the *SQNR*, becomes: $SQNR = 10\log \frac{P_x}{D} = 6.02n + 1.76$ (dB)
- *SQNR* increases exponentially with the number n of bits per sample.
- There is a 6-dB improvement in *SQNR* for each bit added to represent the sample values



$$\Delta = \frac{x_{max} - x_{min}}{L}$$

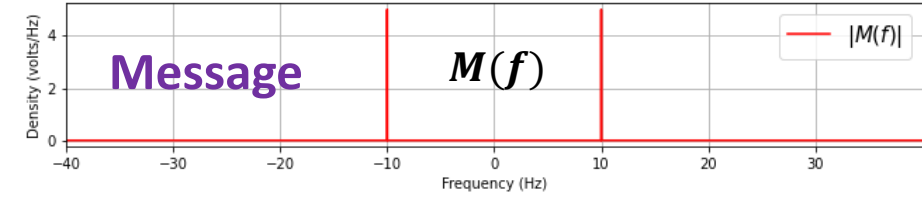
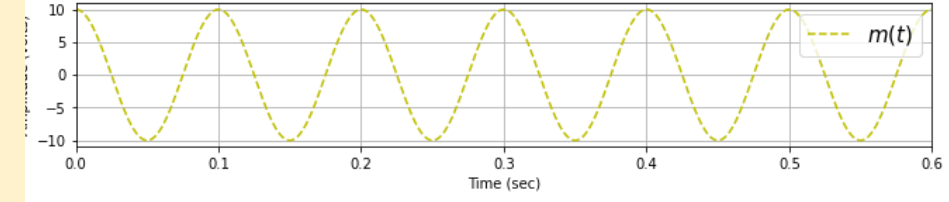
Strong signal applied to a uniform quantizer

Weak signal applied to a uniform quantizer

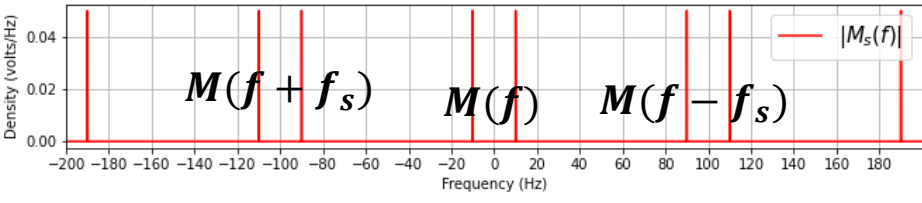
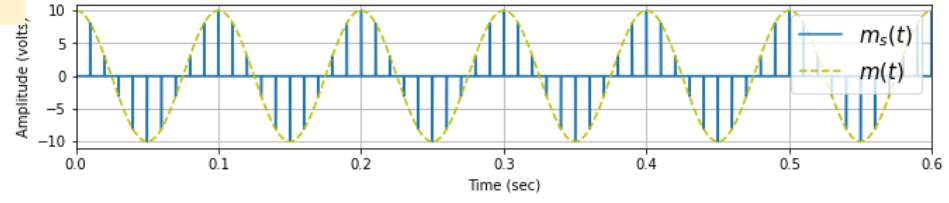
Example: Signal Matches Dynamic Range of a Uniform Quantizer

$x(t) = (10)\cos(2\pi(10)t)$
 Quantizer: (-10, 10);
 $n = 3$ bits (8 levels)
 $\Delta = \frac{20}{8} = 2.5$

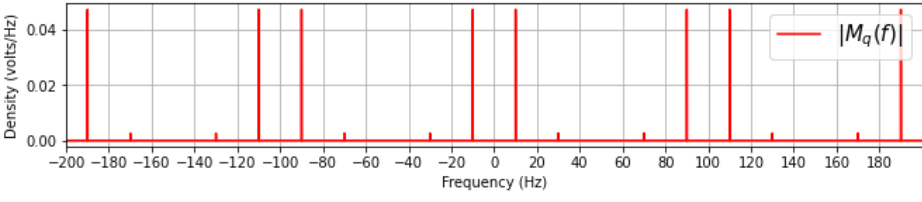
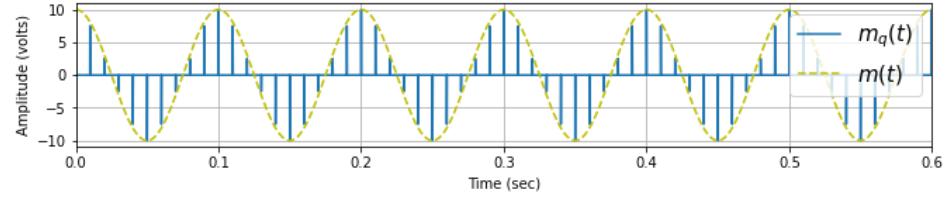
Ideally sampled Signal
 $f_s = 100$



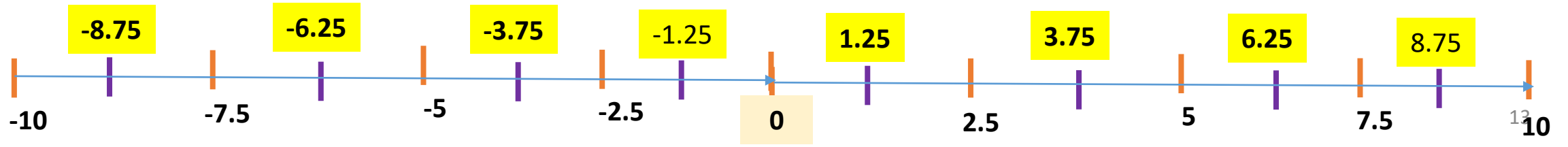
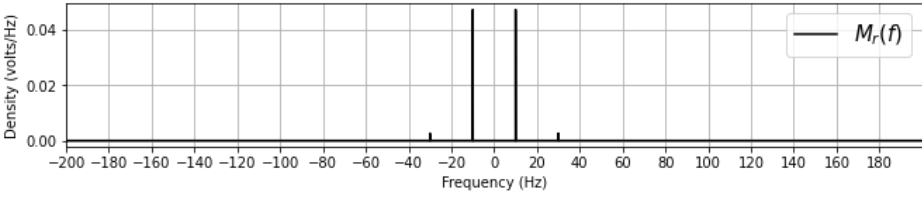
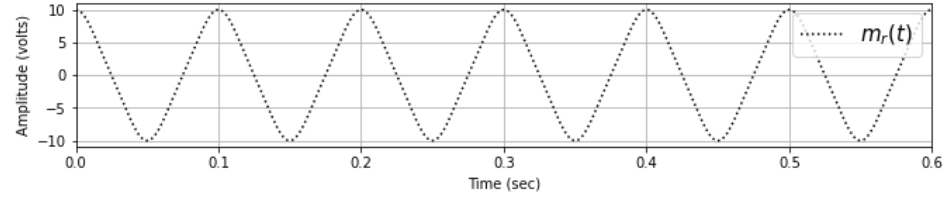
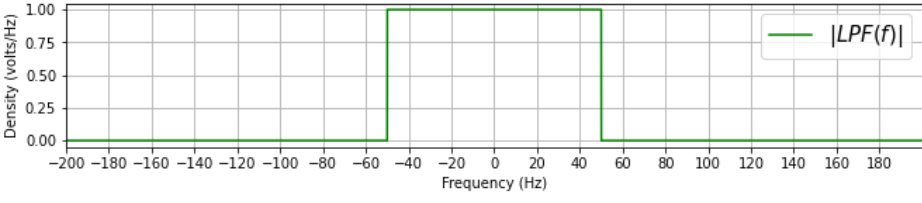
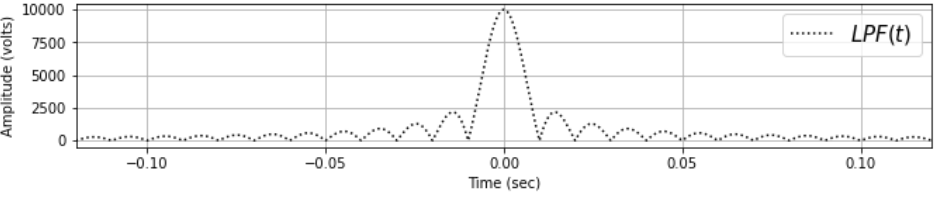
Quantized Signal



Filter B.W = $f_s/2=50$

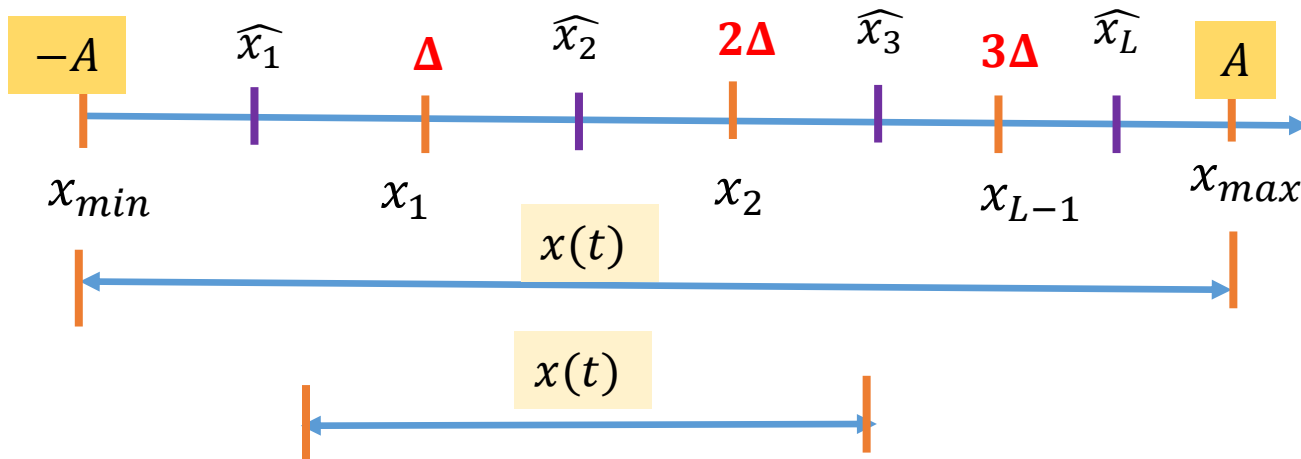


Recovered Signal



Example: Weak Signal Applied to a Uniform Quantizer

- Example:** Now, let the sinusoidal signal $x(t) = A/2 \cos(2\pi f_0 t)$ be applied to the same uniform quantizer of the previous example, with a dynamic range $(-A, A)$. We need to find the $SQNR$.
- Solution:** $SQNR = \frac{(A/2)^2/2}{\Delta^2/12} = \frac{12}{32} L^2 = \frac{12}{32} 2^{2n}$
- In dB, the $SQNR$, becomes: $SQNR = 10 \log \frac{P_x}{D} = 6.02n - 4.77$
- Remark:** If the message signal $x(t)$ is a random signal, with an amplitude probability density function $f_X(x)$ and zero mean ($E(X) = 0$), then the $SQNR$ is given as:
- $SQNR = \frac{E(X^2)}{E(x-\hat{x})^2} = \frac{\int_{-\infty}^{\infty} X^2 f_X(x) dx}{E(x-\hat{x})^2}$; Beyond the scope of this lecture.



$$\Delta = \frac{x_{max} - x_{min}}{L}$$

Strong signal applied to a uniform quantizer

Weak signal applied to a uniform quantizer

Example: Weak Signal Applied to a Uniform Quantizer

$$x(t) = (1)\cos(2\pi(10)t)$$

Quantizer: (-10, 10);

$n = 3$ bits (8 levels)

$$\Delta = \frac{20}{8} = 2.5$$

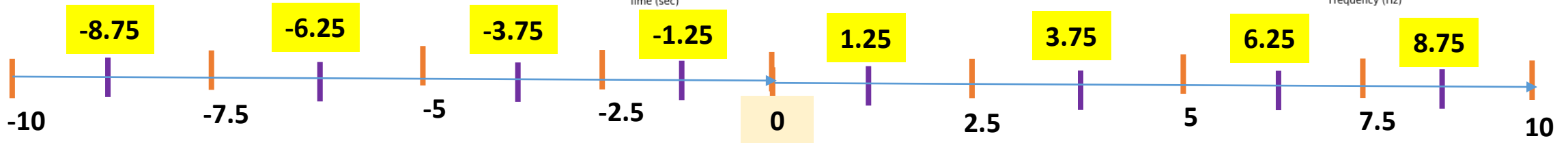
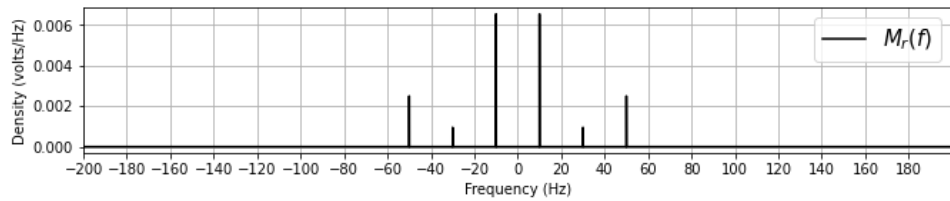
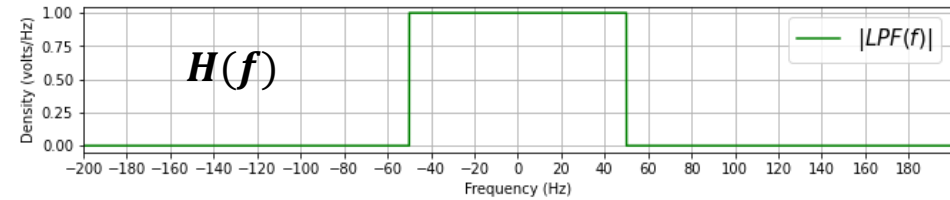
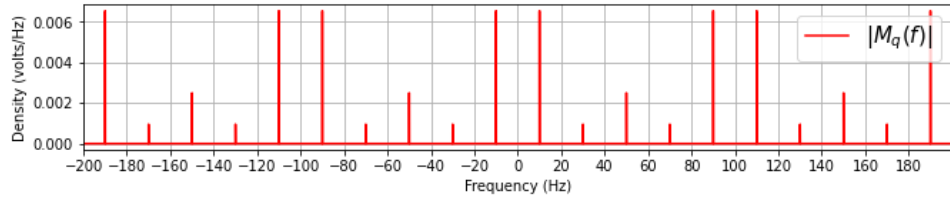
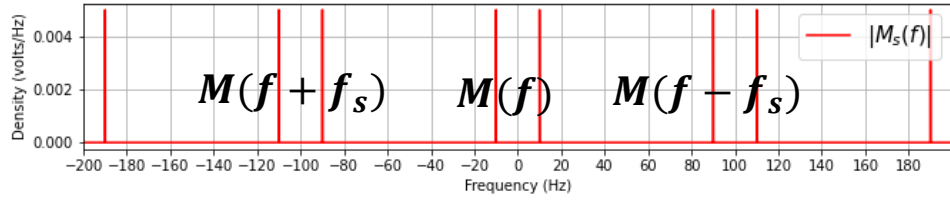
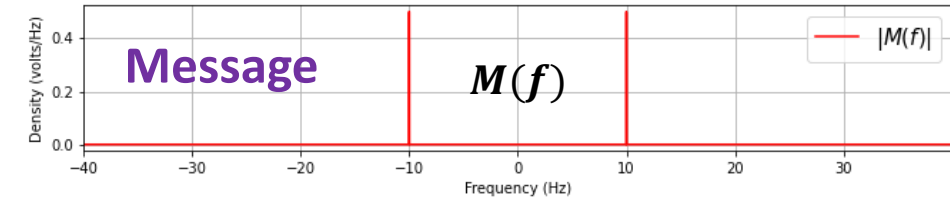
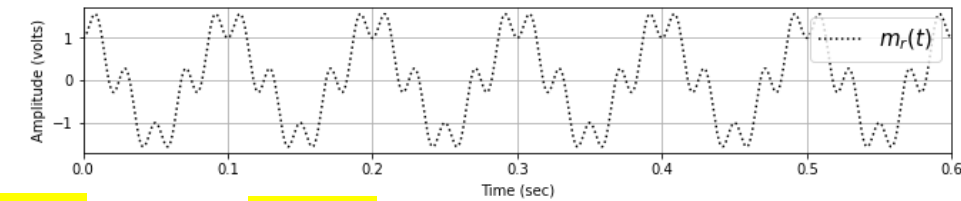
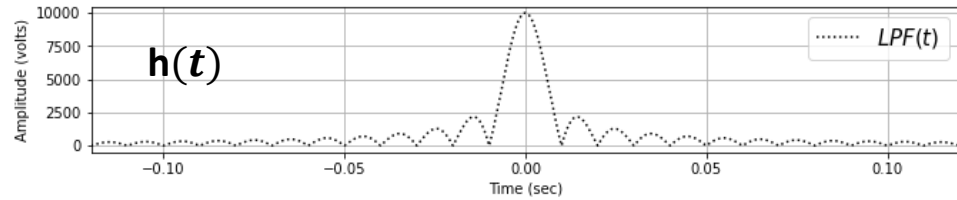
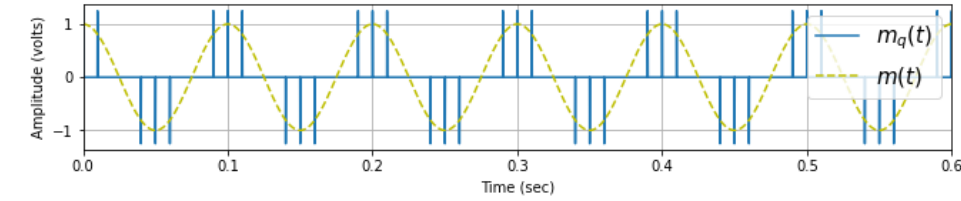
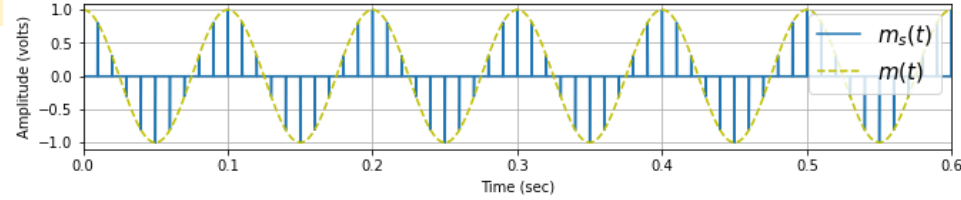
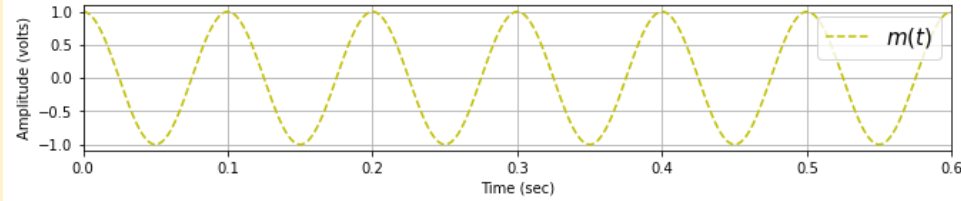
Ideally sampled Signal
 $f_s = 100$

Quantized Signal

$$-\frac{\Delta}{2}, +\frac{\Delta}{2}, +1.25, -1.25$$

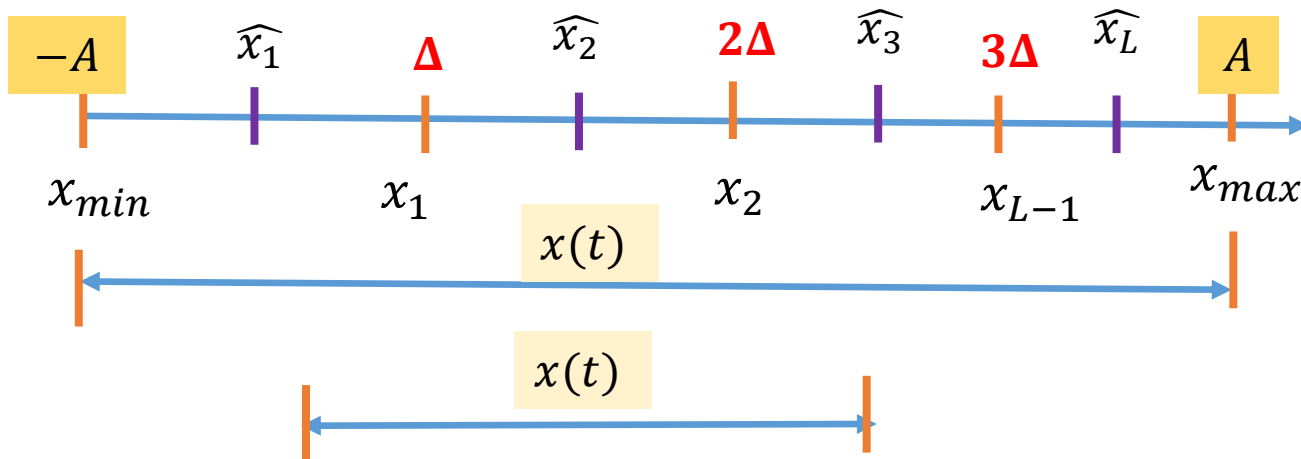
Filter B.W = $f_s/2=50$

Recovered Signal



A Problem with the Uniform Quantizer

- **Problem:** Let the sinusoidal signal $x(t)$ be applied to a uniform quantizer with a dynamic range $(-A, A)$
- We have seen in the previous examples that the SQNR depends on the signal power. As the signal power decreases, the SQNR decreases (quantization noise is constant). **Quality deteriorates**
 - Case 1: $x(t) = A \cos(2\pi f_1 t)$, strong signal applied to the quantizer: $SQNR1 = \frac{3}{2} L^2$; with probability 0.2
 - Case 2: $x(t) = \frac{A}{2} \cos(2\pi f_2 t)$, weak signal applied to the quantizer: $SQNR2 = \frac{12}{32} L^2 = \frac{1}{4} SQNR1$ with probability 0.3
 - Case 3: $x(t) = \frac{A}{4} \cos(2\pi f_3 t)$, weak signal applied to the quantizer: $SQNR3 = \frac{4}{32} L^2 = \frac{1}{16} SQNR1$ with probability 0.5
- The average quantization noise is :
- $E(SQNR) = (0.2) \frac{3}{2} L^2 + (0.3) \frac{12}{32} L^2 + (0.5) \frac{1}{16} L^2 = 0.44375 L^2$
- **Solution:** Use non-uniform quantization (the subject of the next lecture), the use of which will ensure an almost constant SQNR for strong as well as weak signal components.



$$\Delta = \frac{x_{max} - x_{min}}{L}$$

Strong signal applied to a uniform quantizer

Weak signal applied to a uniform quantizer

Non-uniform Robust Quantization

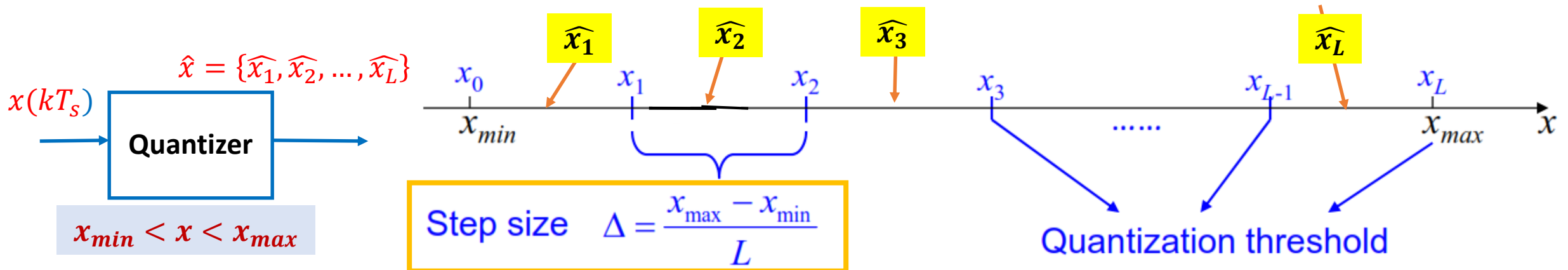
Lecture Outline

- **In this lecture**

- We define the non-uniform robust quantizer
- Define the μ -law compressor and expander characteristics
- Show the improvement in SQNR of non-uniform quantization over uniform quantization.
- Present a number of illustrative examples

Quantization: Basic Definitions

- **Quantization**: is defined as the process of converting the continuous amplitude sample $x(kT_s)$ of a message signal into a discrete amplitude $\hat{x}(kT_s)$ taken from a finite and countable set of L possible values $\{\hat{x}\}$.
- The **dynamic range** of the quantizer is the range of values for which the quantizer is designed, $x_{min} < x < x_{max}$
- This range is partitioned into L intervals such that if $x(kT_s) \in R_i$, the quantizer output will be a level $\hat{x}_i = \{\hat{x}_1, \hat{x}_2, \dots, \hat{x}_L\}$
- The quantizer output is called a **representation or reconstruction level**
- The boundary points separating adjacent regions are called **decision or threshold levels**.
- The spacing between representation levels is called the **step size (Δ)**
- A quantizer is called **uniform** when the L regions are of equal length Δ and the spacing between representation levels is uniform and equals to Δ , where $\Delta = \frac{x_{max} - x_{min}}{L}$.



Drawback of the Uniform Quantizer

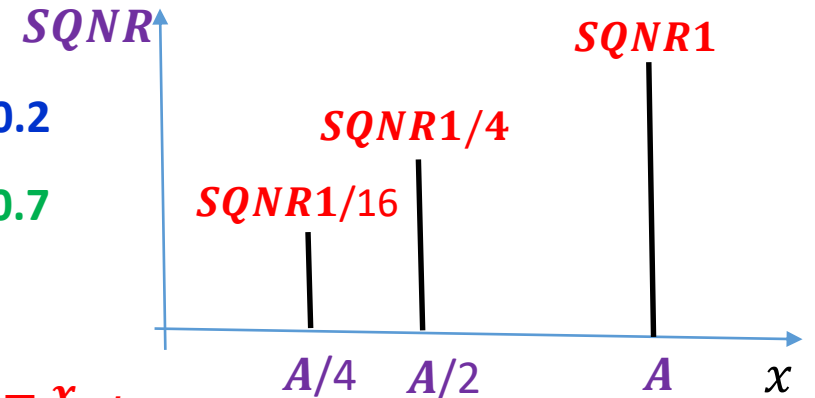
- The Drawback of the uniform quantizer:**

- The uniform quantizer is easy to build; however, it is not optimal when the input signal is weak most of the time. Here, the signal does not use the entire set of available quantization levels.
- Maximum SQNR is achieved when the signal strength matches the dynamic range of the quantizer.

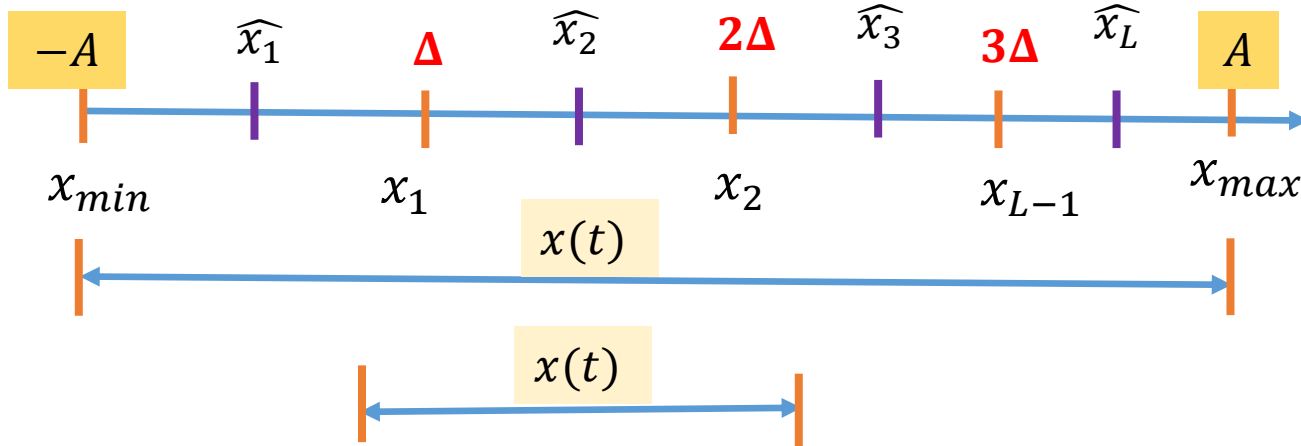
- Demonstration:** Let the sinusoidal signal $x(t)$ be applied to a uniform quantizer with a dynamic range $(-A, A)$

We have seen in the previous lecture that the SQNR depends on the signal power. As the signal power decreases, the SQNR decreases, since the quantization noise is constant ($SQNR = \frac{P_x}{\Delta^2/12}$)

- Let : $x(t) = A \cos(2\pi f_1 t)$, $SQNR1 = \frac{3}{2} L^2$; with probability **0.1**
- $x(t) = \frac{A}{2} \cos(2\pi f_2 t)$, $SQNR2 = \frac{12}{32} L^2 = \frac{1}{4} SQNR1$ with probability **0.2**
- $x(t) = \frac{A}{4} \cos(2\pi f_3 t)$, $SQNR3 = \frac{4}{32} L^2 = \frac{1}{16} SQNR1$ with probability **0.7**



$$\Delta = \frac{x_{max} - x_{min}}{L}$$

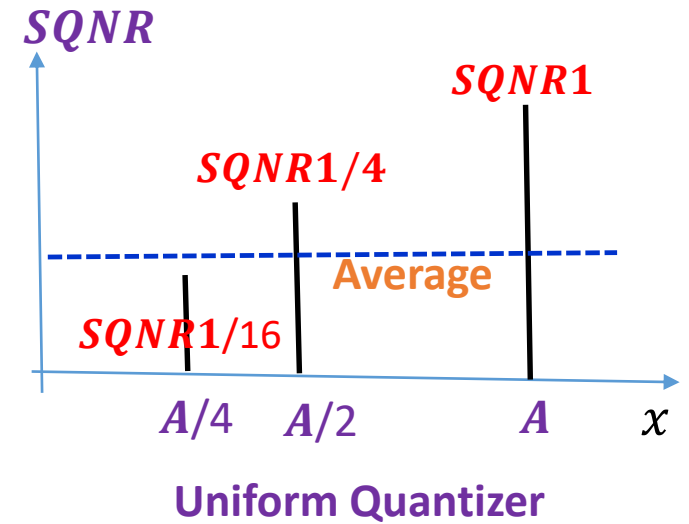


Strong signal applied to a uniform quantizer

Weak signal applied to a uniform quantizer

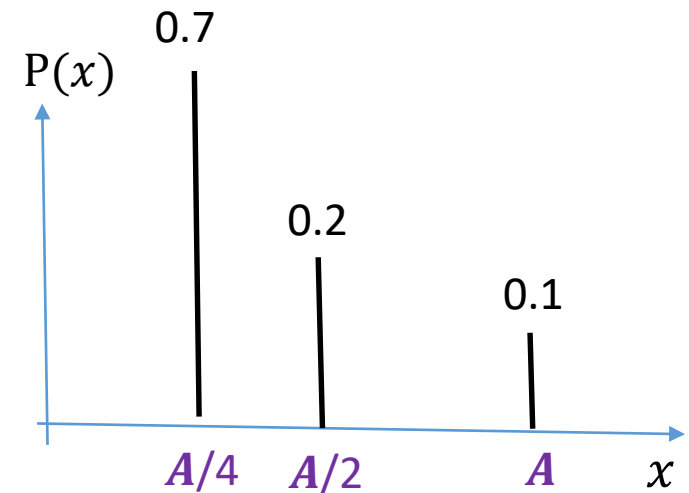
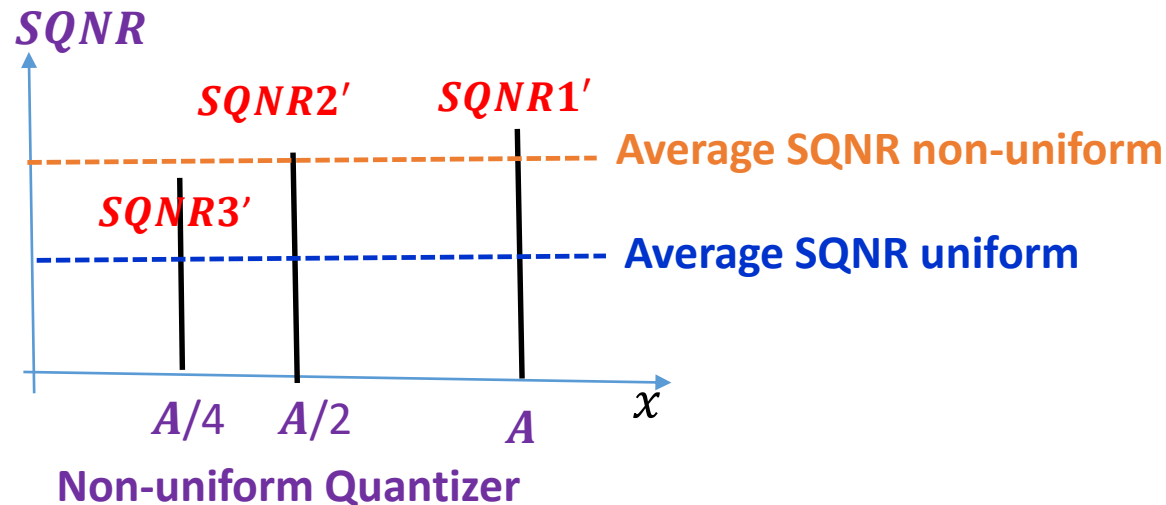
Drawback of the Uniform Quantizer

- The average SQNR is:
- $E(SQNR) = (0.1)SQNR1 + \frac{(0.2)SQNR1}{4} + \frac{(0.7)SQNR1}{16} = (0.19375)SQNR1$
- The average SQNR is closer to that of the smallest value of the SQNR.
- The overall performance is not satisfactory .
- Small amplitudes will be subjected to more distortion than large amplitudes.
- The non-uniform quantization, the subject of this lecture, will ensure an almost constant SQNR for strong as well as weak signal components.



Two Remarks:

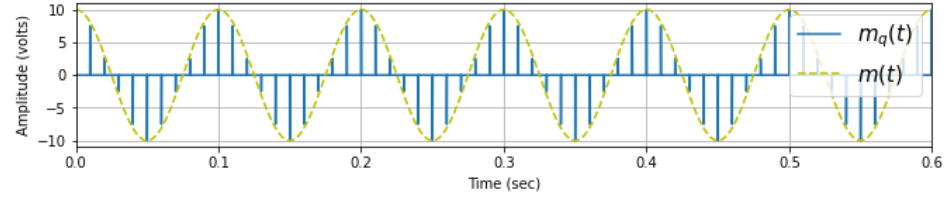
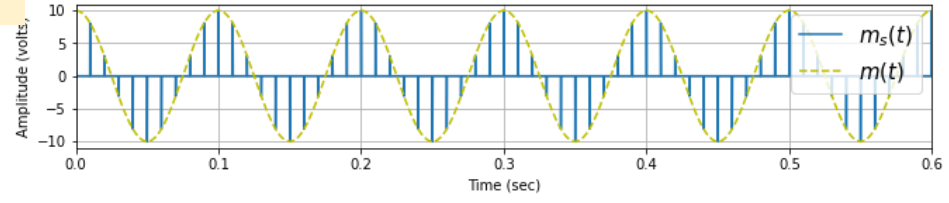
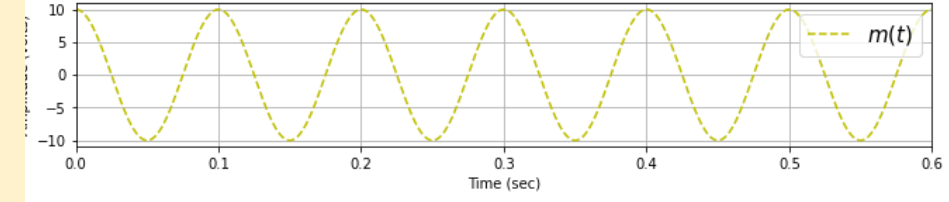
- SQNR is higher than that of the uniform quantizer.
- The SQNR is almost the same for all signal levels



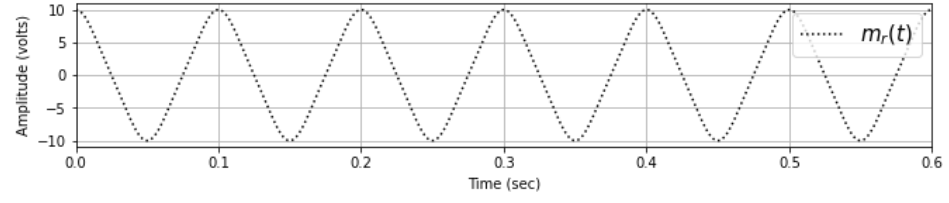
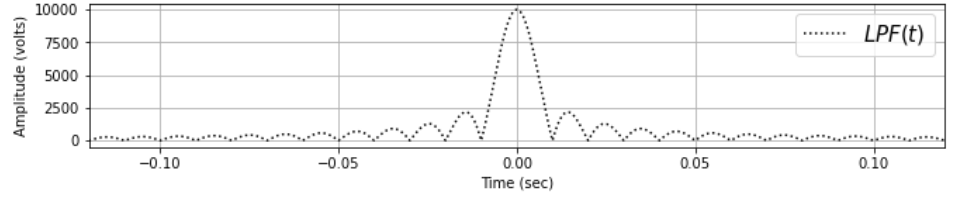
Example: Signal Matches Dynamic Range of a Uniform Quantizer

$x(t) = (10)\cos(2\pi(10)t)$
 Quantizer: (-10, 10);
 $n = 3$ bits (8 levels)
 $\Delta = \frac{20}{8} = 2.5$

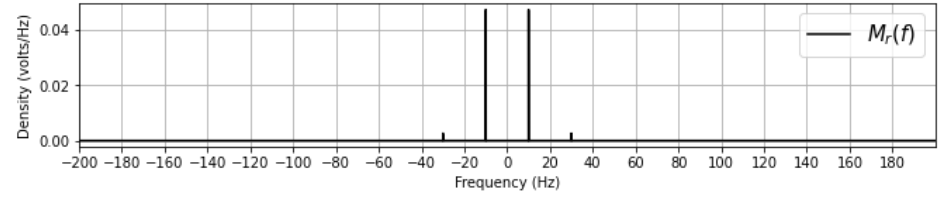
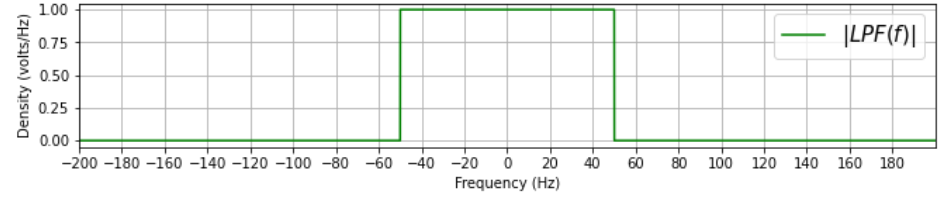
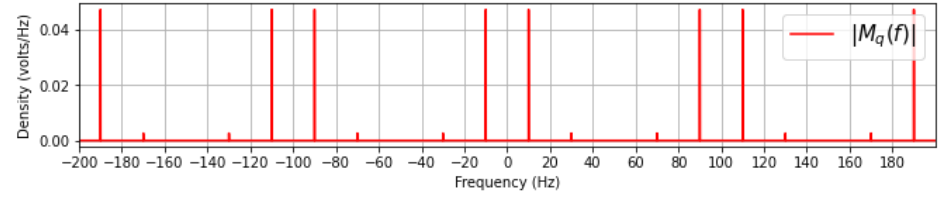
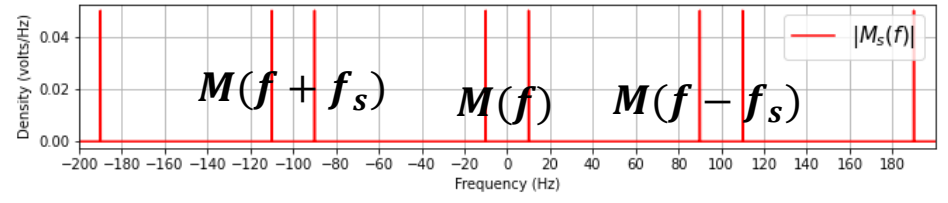
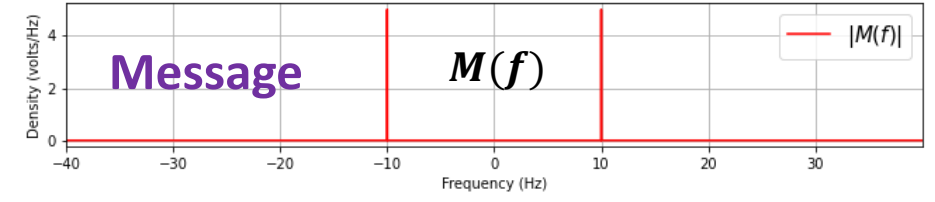
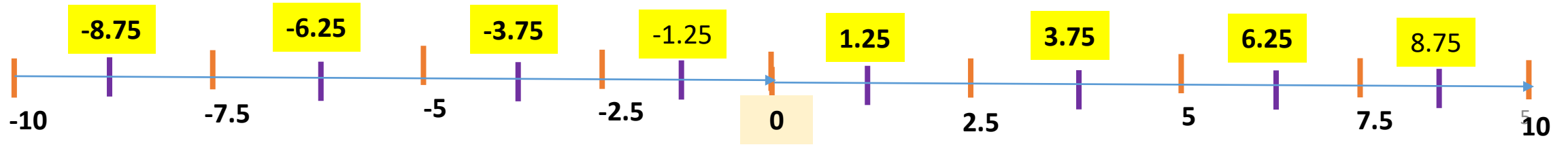
Ideally sampled Signal
 $f_s = 100$



Filter B.W = $f_s/2=50$



All levels are utilized



Example: Weak Signal Applied to a Uniform Quantizer

$$x(t) = (1)\cos(2\pi(10)t)$$

Quantizer: (-10, 10);

$n = 3$ bits (8 levels)

$$\Delta = \frac{20}{8} = 2.5$$

Ideally sampled Signal
 $f_s = 100$

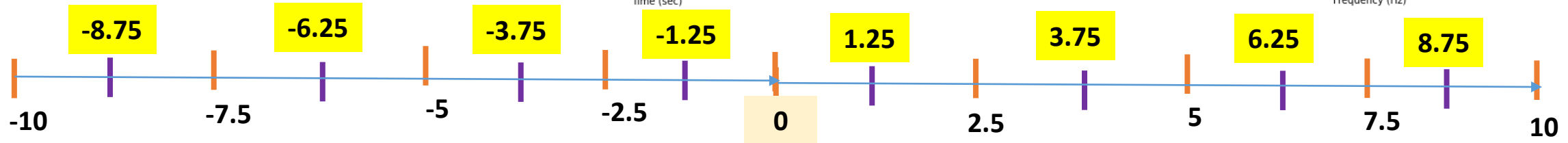
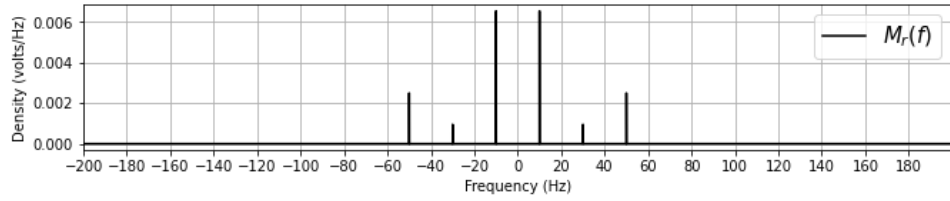
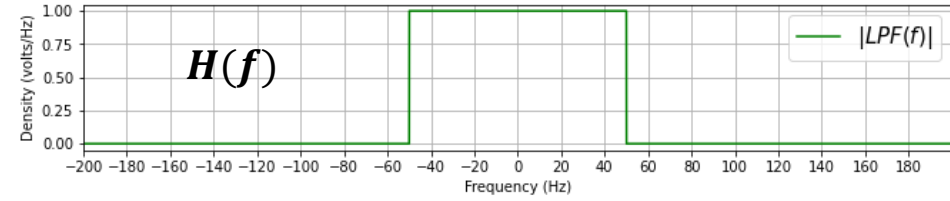
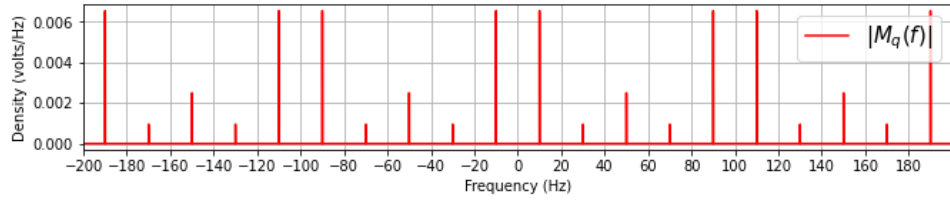
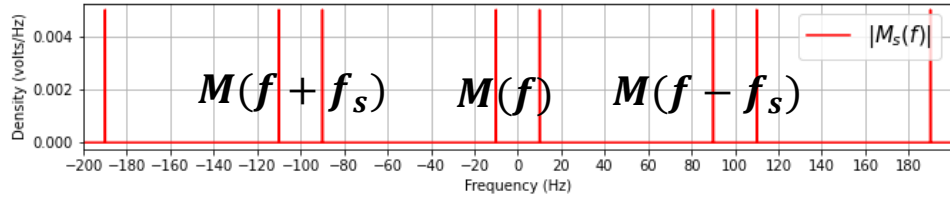
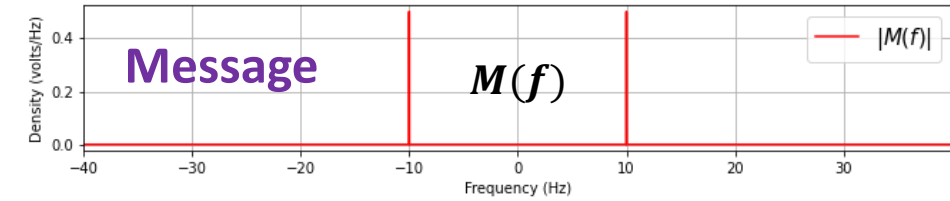
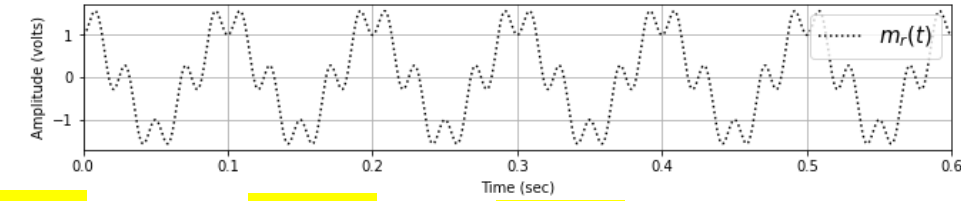
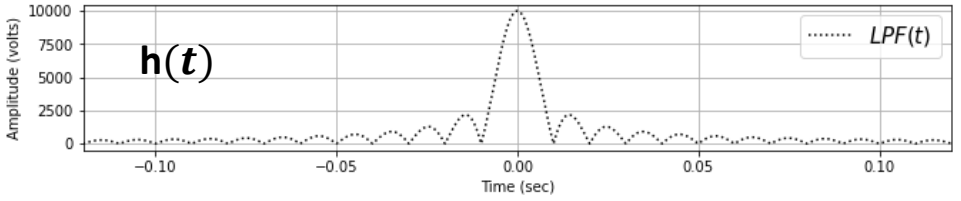
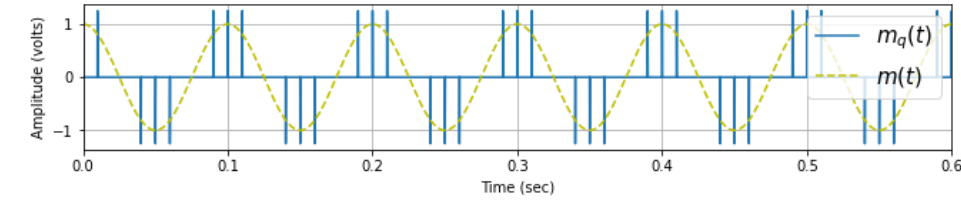
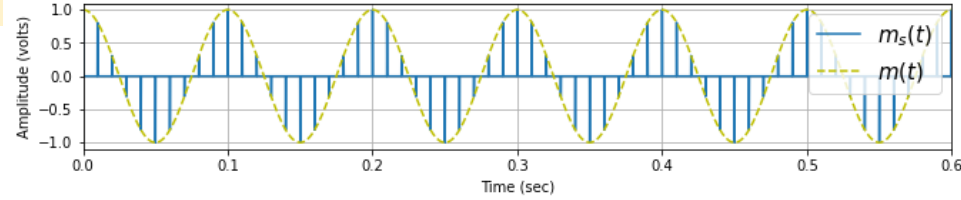
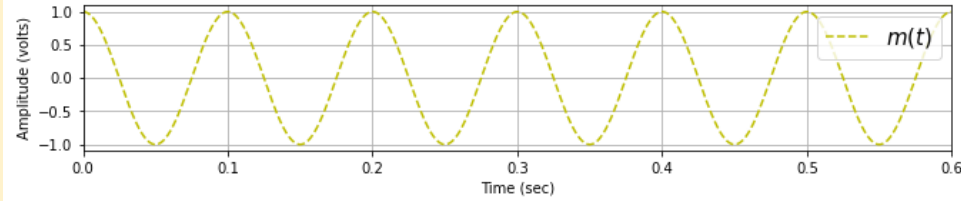
Quantized Signal

$$-\frac{\Delta}{2}, +\frac{\Delta}{2}, +1.25, -1.25$$

Filter B.W = $f_s/2=50$

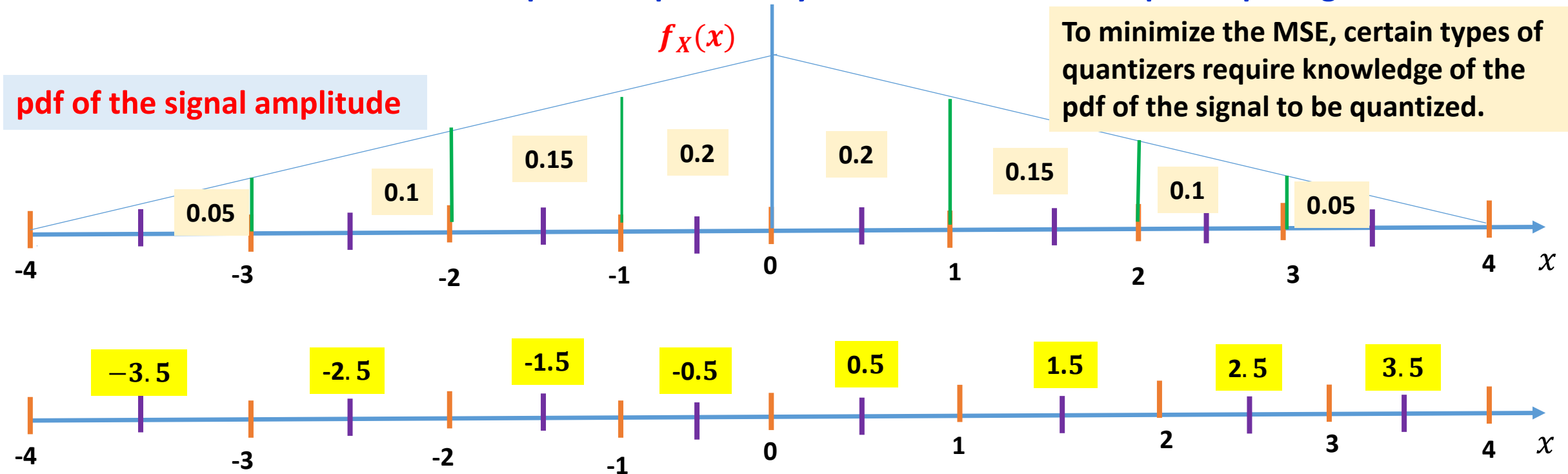
Recovered Signal

Only two levels are utilized



Uniform and Non-uniform Robust Quantization

- In practical applications, especially speech signals, small signal amplitudes occur more often than large signal amplitudes. This means that for the same signal, small amplitudes will be subject to distortion more than large amplitudes. **That is, larger amounts of distortion have a higher probability of occurrence.**
- The non-uniform robust quantizer, which employs the μ -law can
 - Provide a SQNR that is somewhat constant and independent of the signal strength
 - Provide a SQNR that is also independent probability distribution of the sampled input signal

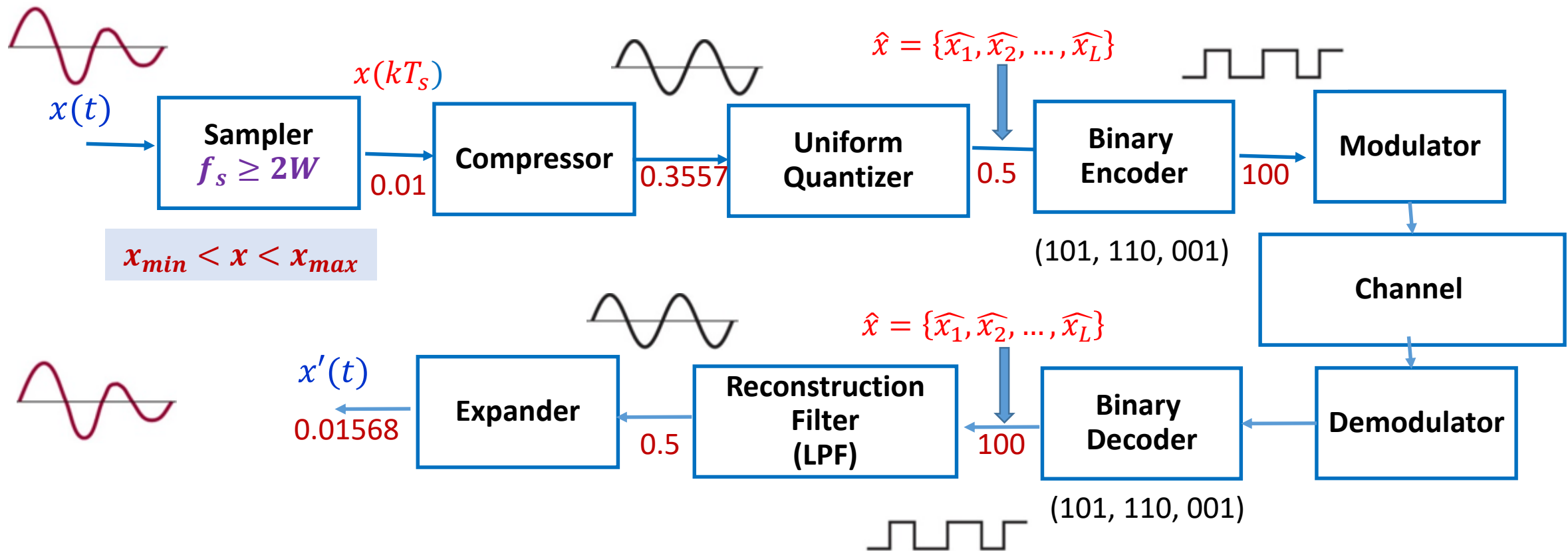


Non-uniform Robust Quantization

- We will use a type of non-uniform quantizers, called **companding**, that **does not require knowledge of the pdf of the signal to be quantized, and yields an almost uniform SQNR over a wide range of signal variations.**
- The process of pre-distorting the signal at the transmitter is known as (signal) **compression**. At the receiver, this process is reversed to remove distortion and is known as (signal) **expansion**. The two operations together, are typically, referred to as **companding** (or compansion).
- The compressor is a nonlinear operation that amplifies weak signal values more than it amplifies large signal values, thus stretching the signal over more representation levels. This will enhance weak signal levels and improve their SQNR. Large signal levels will also suffer from distortion, but the overall effect on the signal is an improvement in SQNR.
- Since the probability of smaller amplitudes is higher than the larger amplitudes, the overall result is an improvement.
- In North America, μ -law companding (with $\mu = 255$) is the standard.

Robust (Non-uniform) Quantization

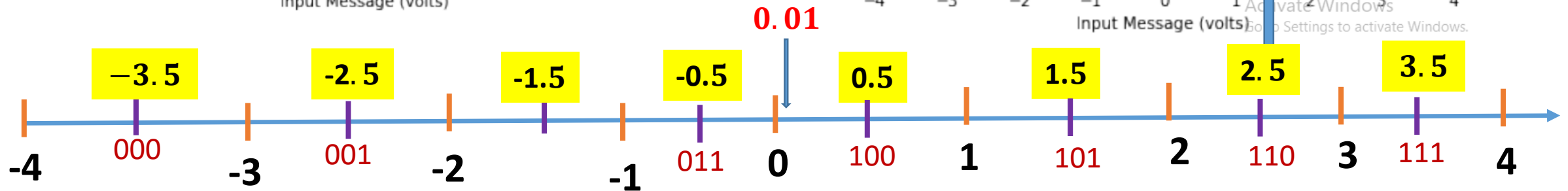
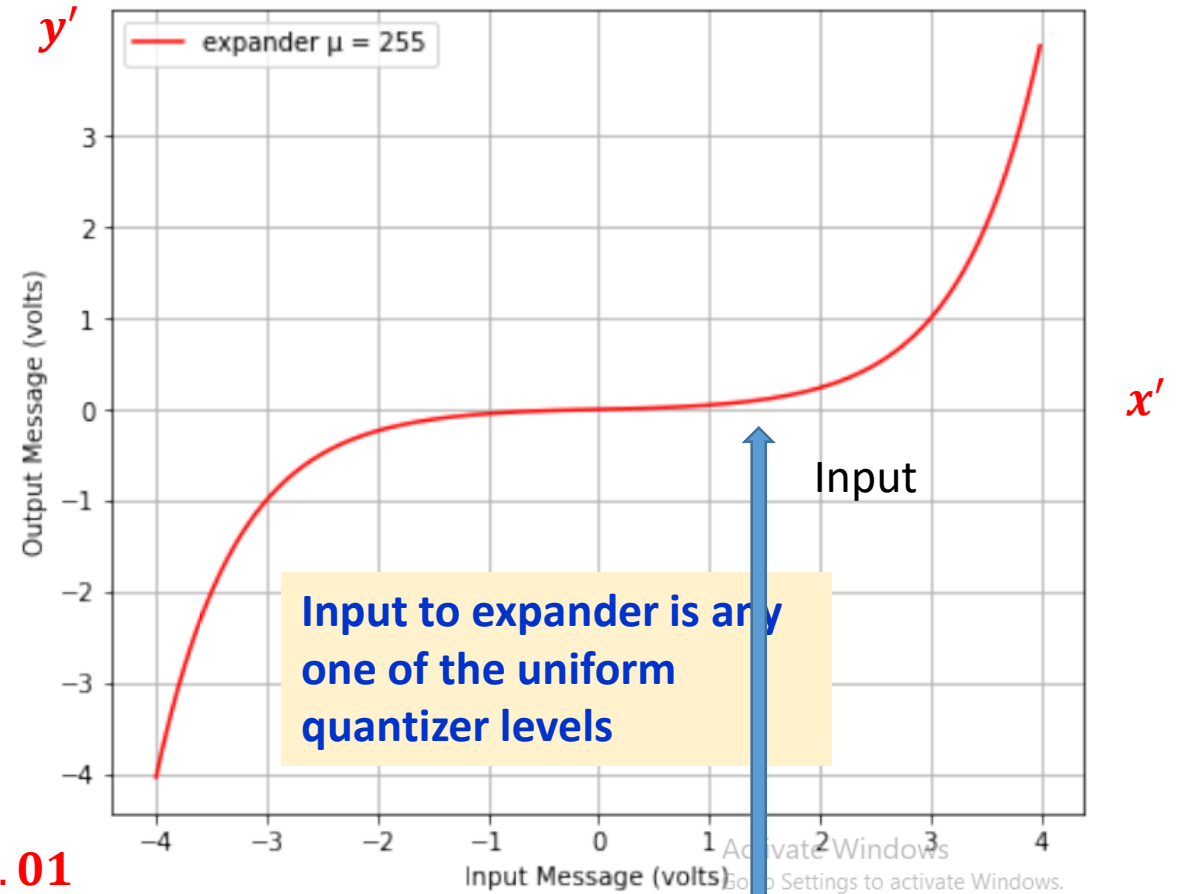
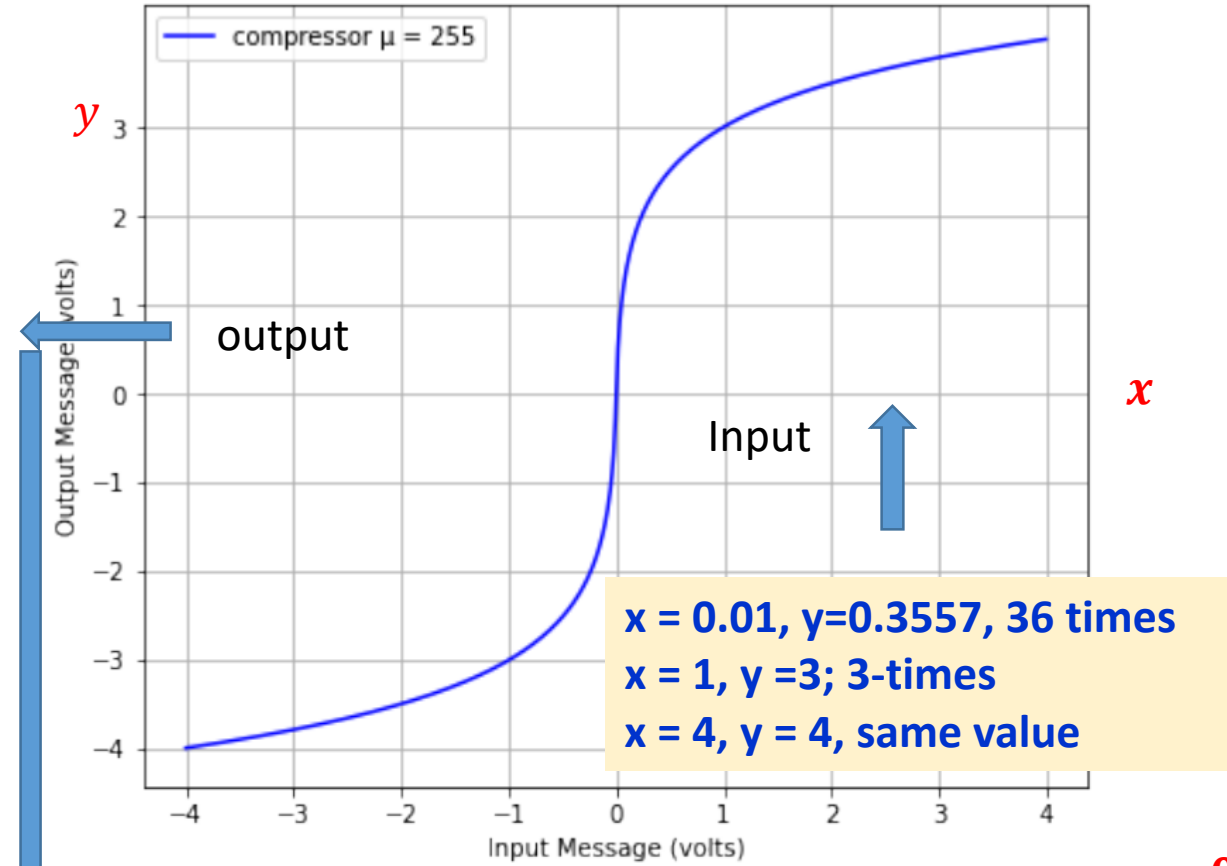
- In summary, companding is performed as follows:
 - Compress the signal using the μ -law. The output is approximately uniformly distributed.
 - Apply the compressed sample to a uniform quantizer
 - Transmit the quantized sample to the receiver.
 - Apply the received sample to the expander. The output is the desired signal value.



Non-uniform Robust Quantization

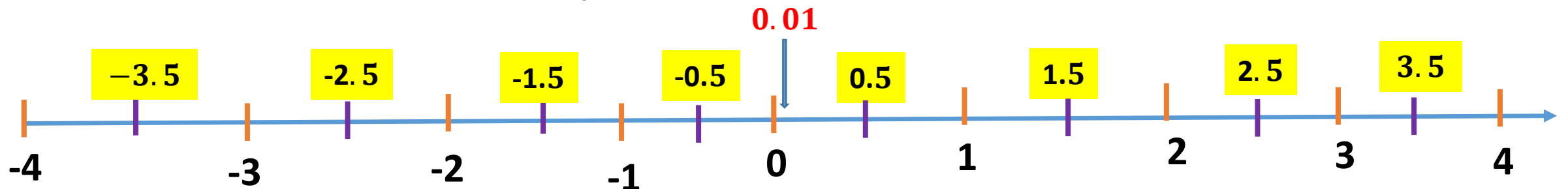
$$y = y_{max} \frac{\ln \left[1 + \mu \left(\frac{|x|}{x_{max}} \right) \right]}{\ln(1 + \mu)} \operatorname{sgn}(x)$$

$$\frac{y'}{y'_{max}} = \left[\exp \left(\frac{x'}{x'_{max}} (\ln(1 + \mu)) \right) - 1 \right] / \mu \operatorname{sgn}(x')$$



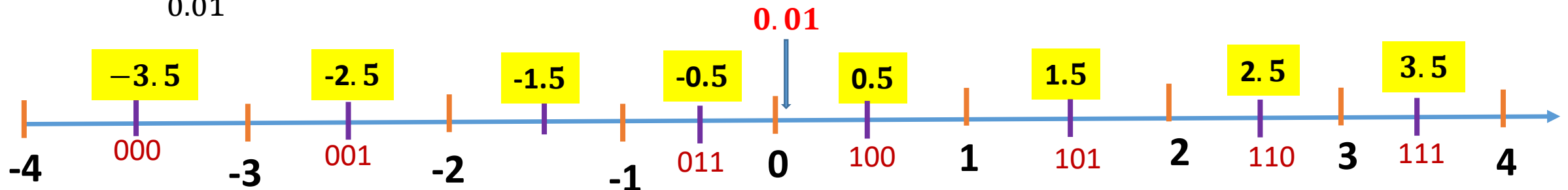
Non-uniform Robust Quantization

- **Example:** If a sample of magnitude 0.01 is applied to a 3-bit uniform quantizer with a dynamic range (-4, 4) V. Find the quantizer output, the quantization error, and the value of the sample at the receiving side.
- If the sample is applied to quantizer that employs companding, find the value of the received sample, and the quantization error.
- **Uniform Quantization:** when $x = 0.01$, $\hat{x} = 0.5$; This is the same voltage that is supposed to be received, assuming no transmission error.
- The error is: $|x - \hat{x}| = |0.01 - 0.5| = 0.49$
- The % of error is: $\hat{x} = \frac{0.5 - 0.01}{0.01} 100\% = 4900\%$.
- The error is 49 times the sample value.



Non-uniform Robust Quantization

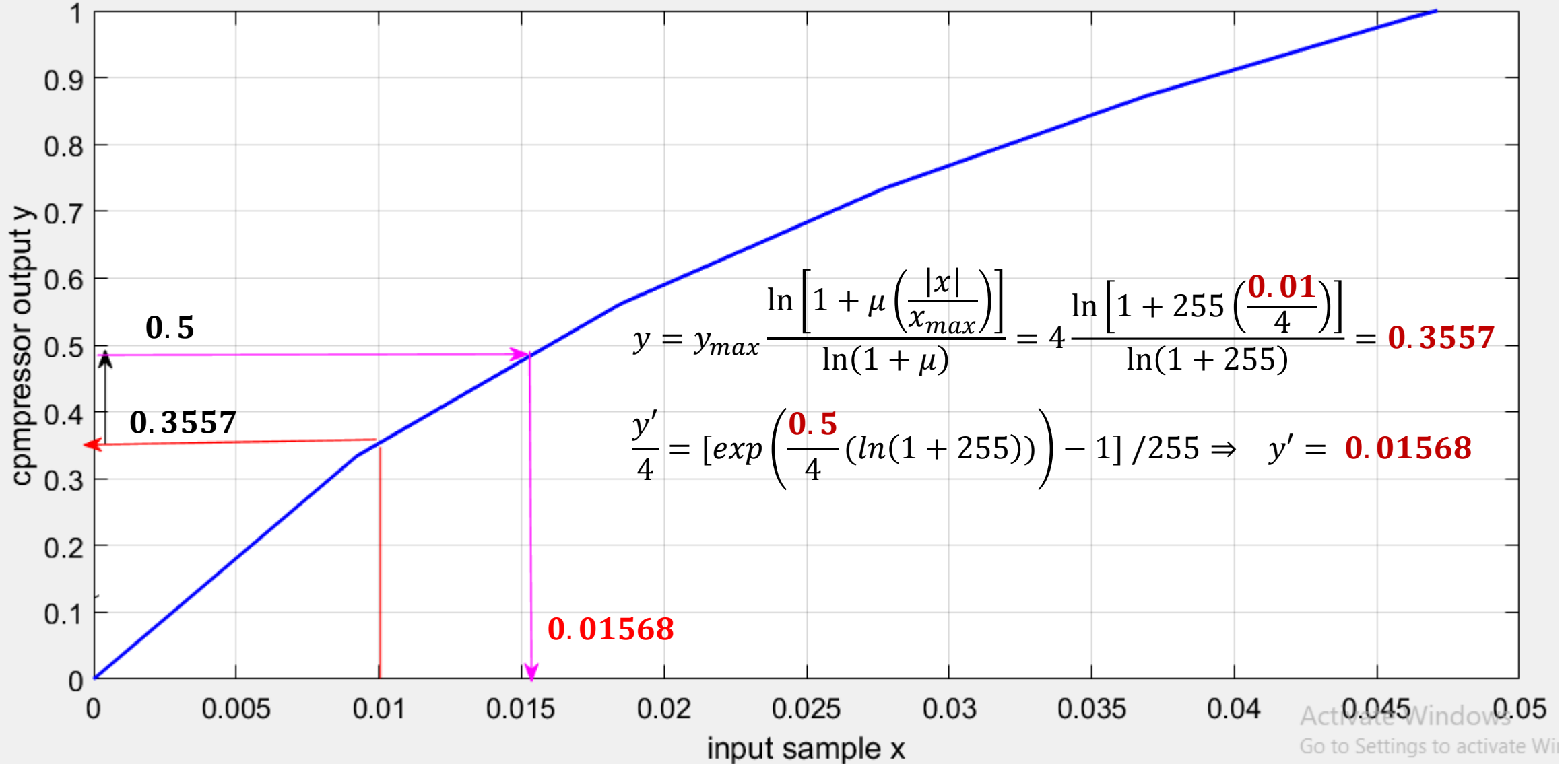
- **Non-uniform Quantization**: when $x = 0.01$ is applied to the μ -law,
- $y = y_{max} \frac{\ln\left[1+\mu\left(\frac{|x|}{x_{max}}\right)\right]}{\ln(1+\mu)} = 4 \frac{\ln\left[1+255\left(\frac{0.01}{4}\right)\right]}{\ln(1+255)} = \mathbf{0.3557}$
- The 0.3557 is applied to the uniform quantizer, whose output is **0.5**.
- The 0.5 V is encoded as 100 and transmitted. Assume no transmission errors.
- At the receiver, the 0.5 is applied to the expander (the inverse μ -law characteristic)
- $\frac{y'}{4} = \left[\exp\left(\frac{0.5}{4} (\ln(1 + 255))\right) - 1\right] / 255 \Rightarrow y' = \mathbf{0.01568}$
- Received sample $y' = \mathbf{0.01568}$
- The error is: $|x - y'| = |\mathbf{0.01} - \mathbf{0.0157}| = \mathbf{0.0057}$
- Error: $\frac{0.01-0.0157}{0.01} 100\% = \mathbf{57\%}$



$$y = y_{max} \frac{\ln \left[1 + \mu \left(\frac{|x|}{x_{max}} \right) \right]}{\ln(1 + \mu)} \operatorname{sgn}(x)$$

$$\frac{y'}{y'_{max}} = \left[\exp \left(\frac{x'}{x'_{max}} (\ln(1 + \mu)) \right) - 1 \right] / \mu$$

mu law characteristic



$$y = y_{max} \frac{\ln \left[1 + \mu \left(\frac{|x|}{x_{max}} \right) \right]}{\ln(1 + \mu)} = 4 \frac{\ln \left[1 + 255 \left(\frac{0.01}{4} \right) \right]}{\ln(1 + 255)} = 0.3557$$

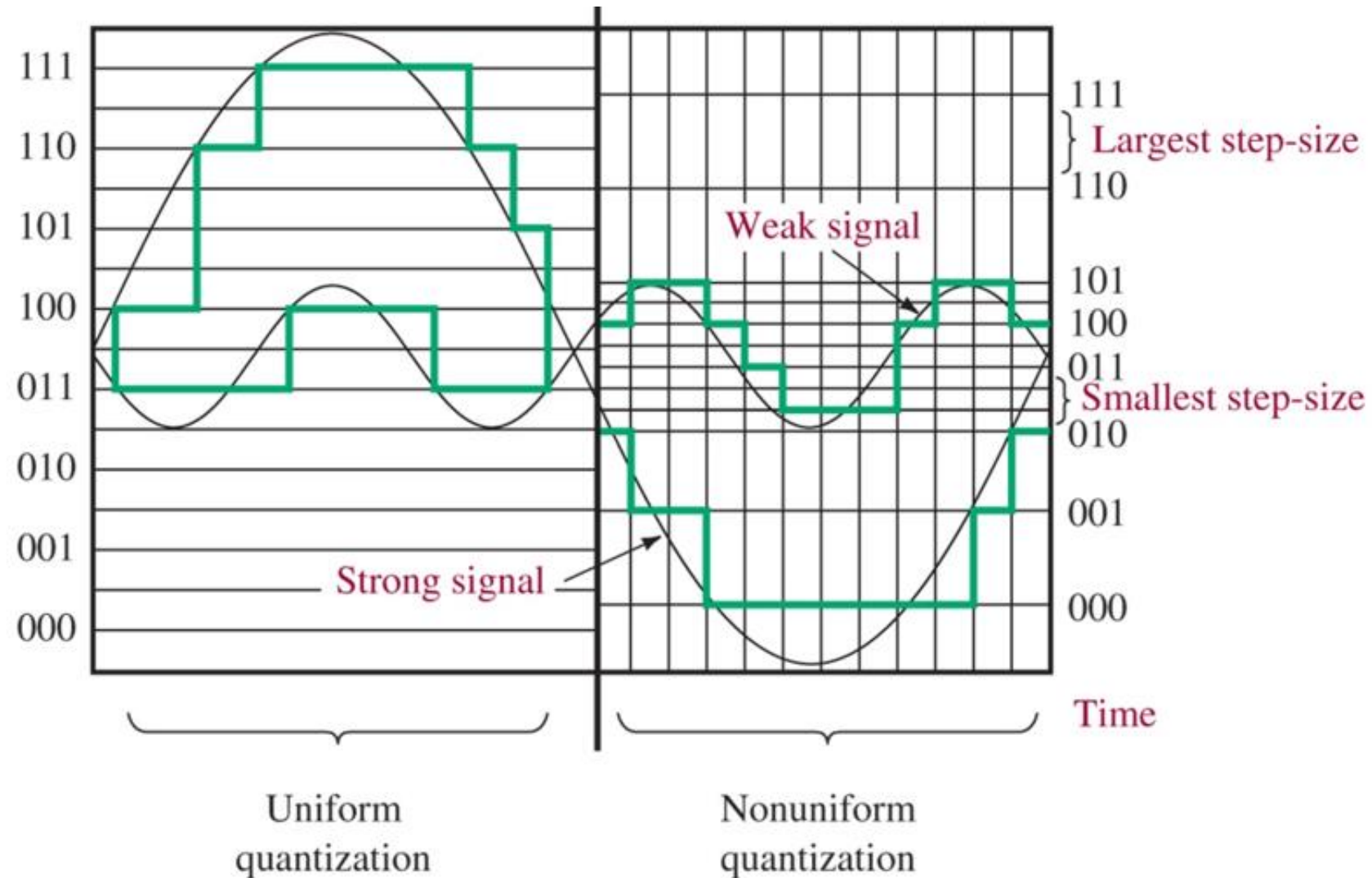
$$\frac{y'}{4} = \left[\exp \left(\frac{0.5}{4} (\ln(1 + 255)) \right) - 1 \right] / 255 \Rightarrow y' = 0.01568$$

Uniform and Non-uniform Robust Quantization: More Values

Sample Value	Uniform Quantizer		Non-uniform Quantizer	
	Output	% Error	Output	% Error
0.01	0.5	$\frac{0.01 - 0.5}{0.01} 100\% = \mathbf{4900\%}$	0.0157	$\frac{0.01 - 0.0157}{0.01} 100\% = \mathbf{57\%}$
0.07	0.5	$\frac{0.07 - 0.5}{0.07} 100\% = \mathbf{707\%}$	0.1098	$\frac{0.07 - 0.1098}{0.07} 100\% = \mathbf{56.8\%}$
1.7	1.5	$\frac{1.7 - 1.5}{1.7} 100\% = \mathbf{11.76\%}$	1.9922	$\frac{1.7 - 1.9922}{1.7} 100\% = \mathbf{17.18\%}$
2.9	2.5	$\frac{2.9 - 2.5}{2.9} 100\% = \mathbf{13.79\%}$	1.9922	$\frac{2.9 - 1.9922}{2.9} 100\% = \mathbf{31.3\%}$
3.8	3.5	$\frac{3.8 - 3.5}{3.8} 100\% = \mathbf{7.89\%}$	1.9922	$\frac{3.8 - 1.9922}{3.8} 100\% = \mathbf{47.5\%}$

Uniform and Non-uniform Robust Quantization

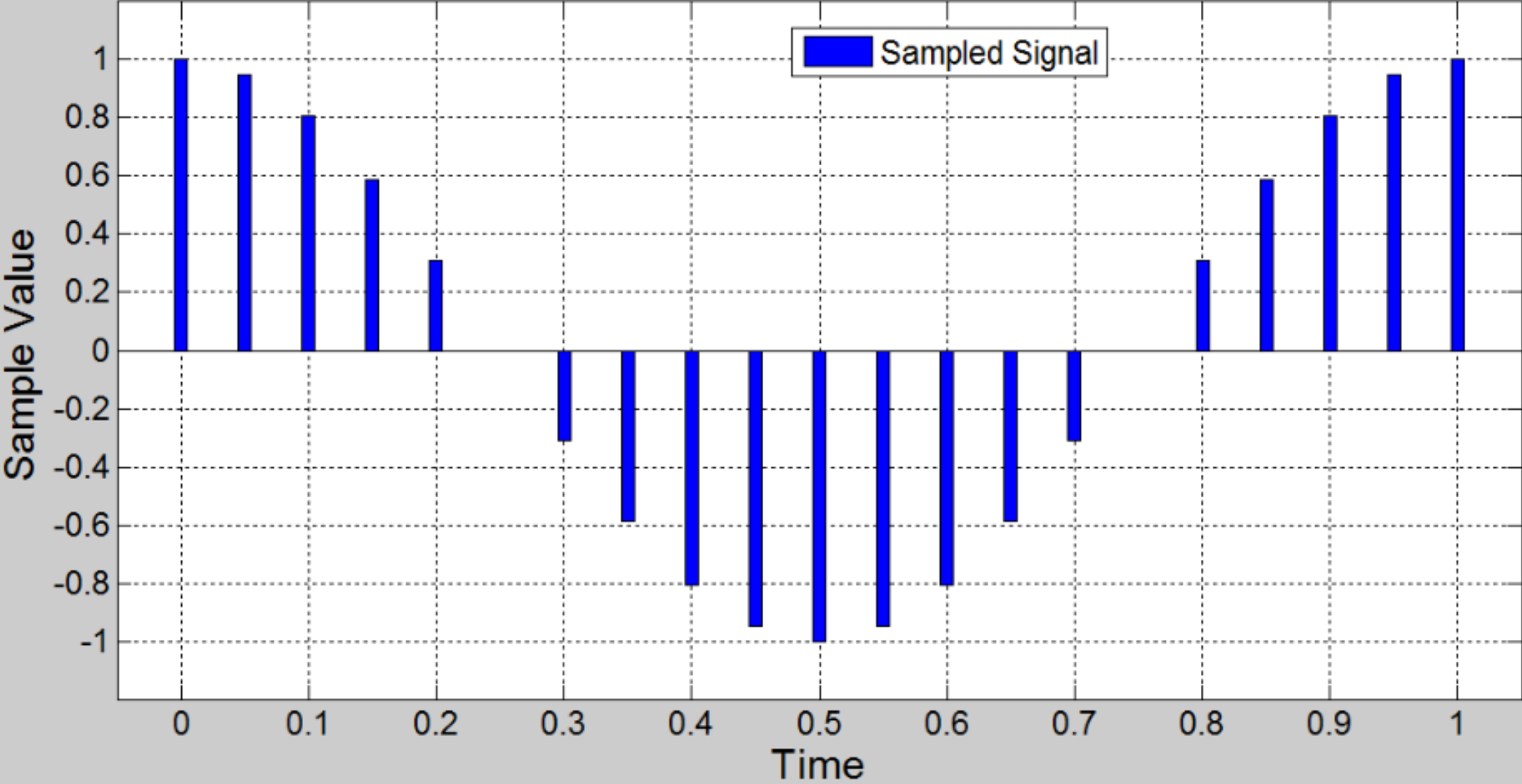
- This figure shows a weak and a strong signal applied to a uniform and a non-uniform quantizer. As we can see, the non-uniform quantizer represents both signals quite adequately.



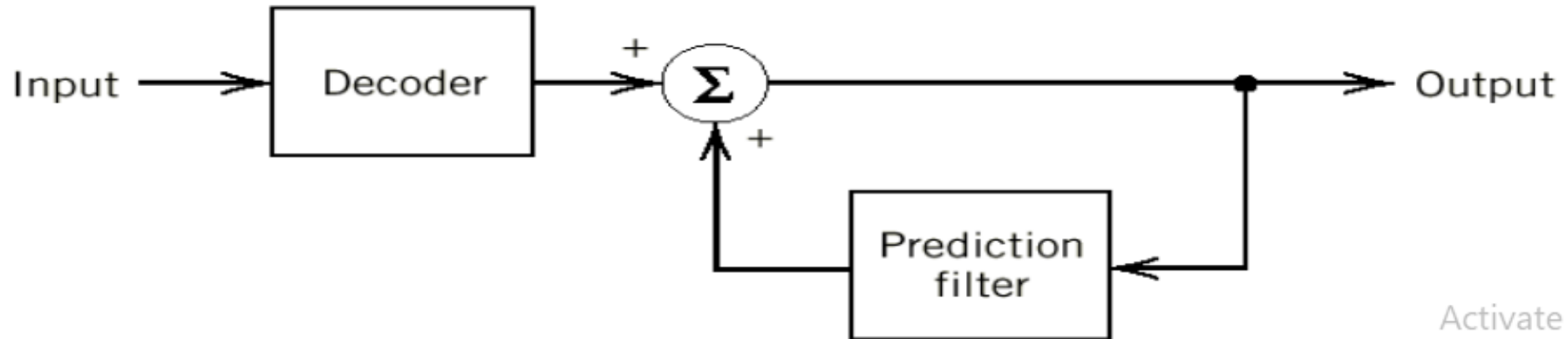
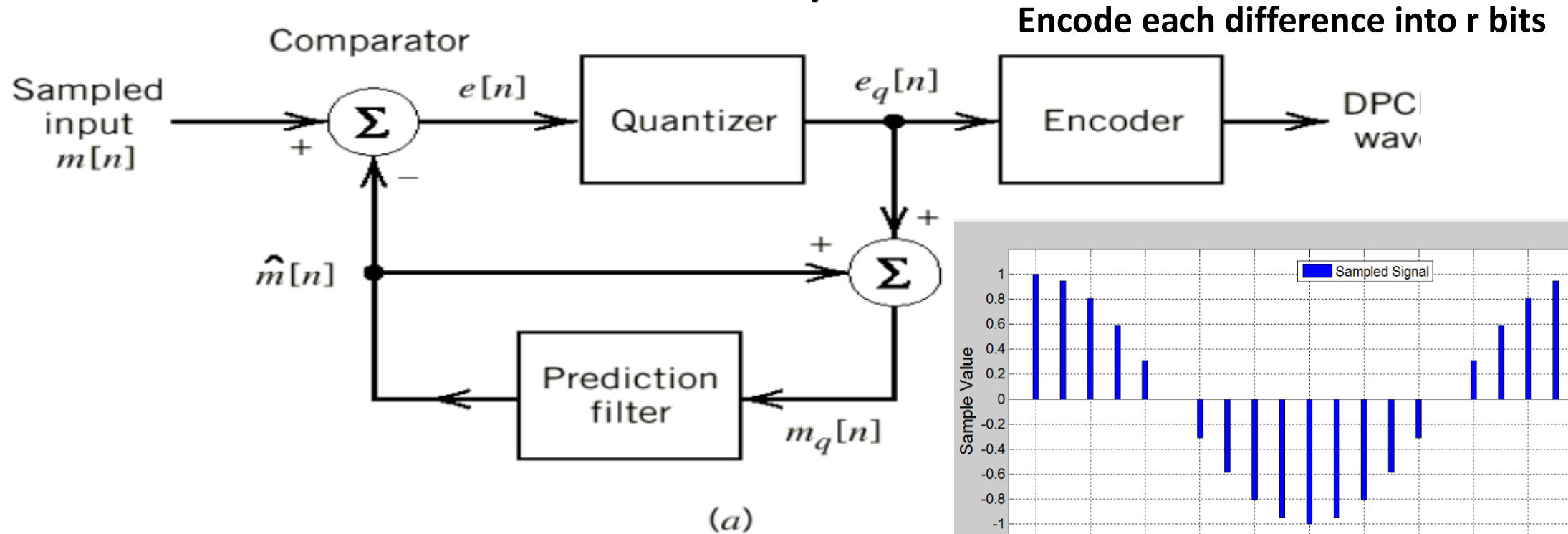
Differential Pulse Code Modulation

- The quantizers, that we studied so far, are memoryless, in the sense that quantization is done on a sample-by-sample basis. Each sample is quantized and encoded into n binary digits, regardless of any correlation with other samples.
- A ***differential pulse-code modulation (DPCM)*** quantizer quantizes the difference between a sample and a predicted value of that sample. Here, correlation between successive samples is utilized.
- The prediction is based, in general, on past m samples of the signal. If successive samples are highly correlated, the predictor output will be very close to the next sample value, and hence the prediction error will be small.
- An error with a small variance further means that **fewer bits ($r < n$) are needed to represent the error.**
- At the receiver, a predictor similar to the one used at the transmitter is used to reconstruct the original waveform

Differential Pulse Code Modulation



DPCM: Basic Operation



DPCM: Linear Prediction Filter

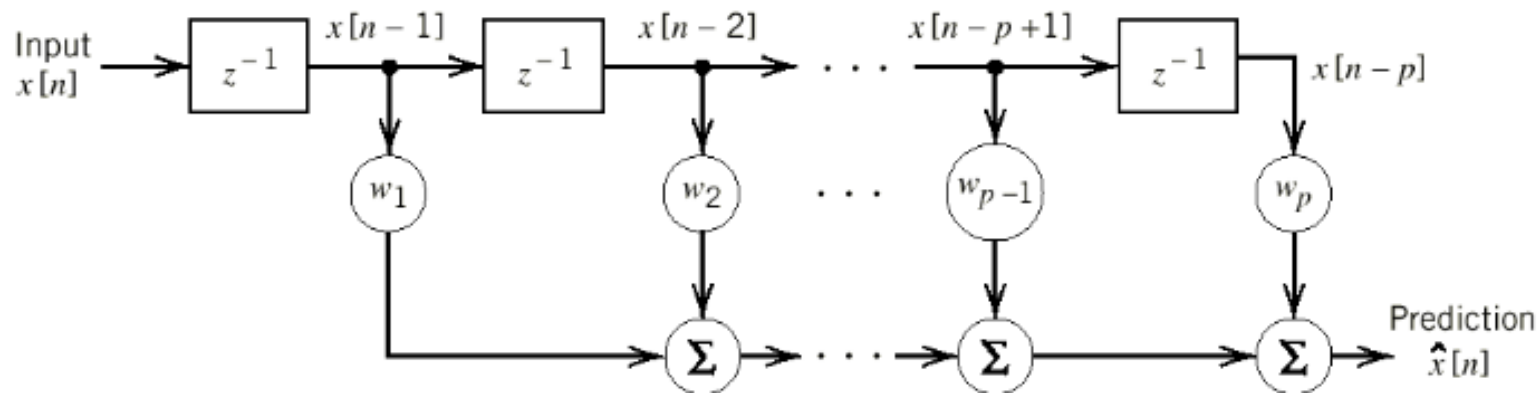
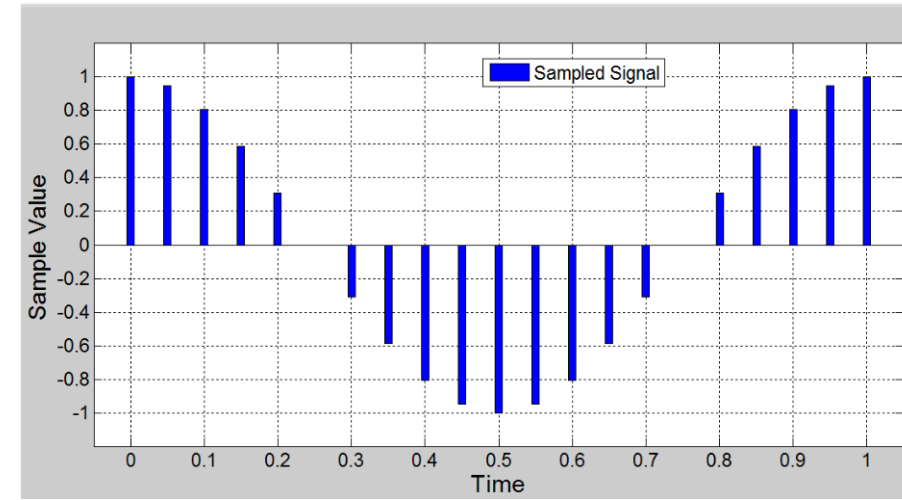
It is a discrete-time, finite-duration impulse response filter (FIR), which consists of three blocks:

1. Set of p (p : prediction order) unit-delay elements (z^{-1})
2. Set of multipliers with coefficients w_1, w_2, \dots, w_p
3. Set of adders (Σ)

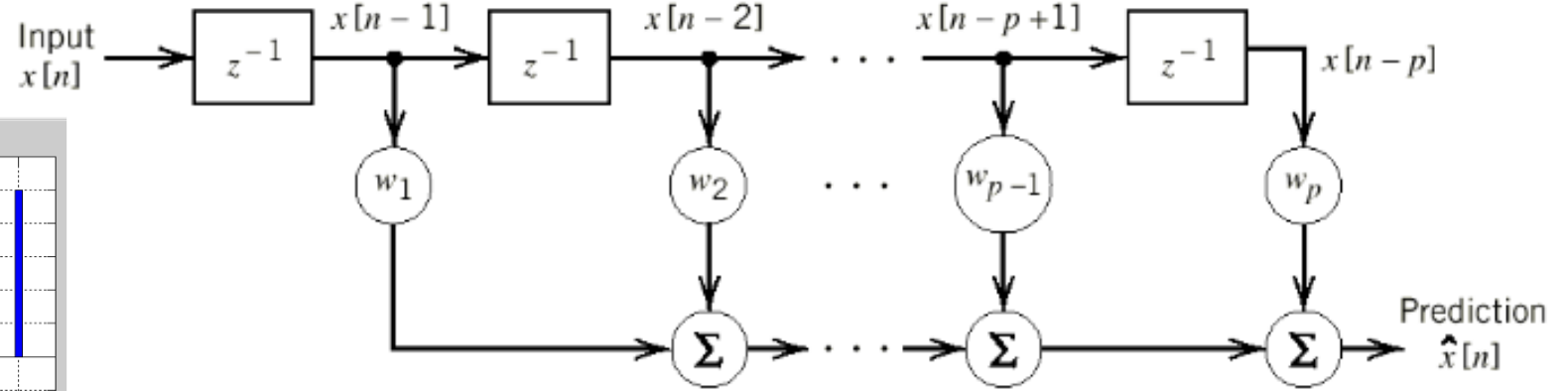
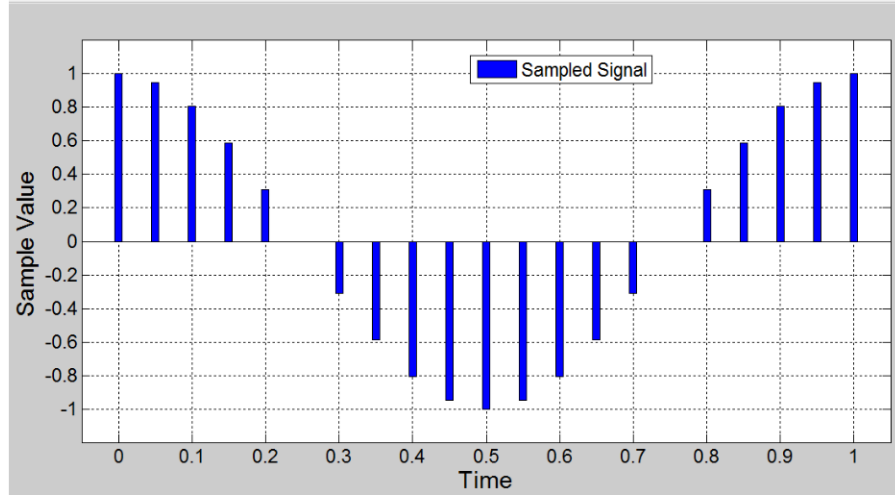
This filter expresses the predicted value of the sample at time (nT_s) as a linear combination of the past p samples of the signal.

$$\hat{x}(n) = w_1x(n - 1) + w_2x(n - 2) + \dots + w_px(n - p)$$

The coefficients w_1, w_2, \dots, w_p are chosen so as to minimize the mean square error $E(x(n) - \hat{x}(n))^2$.



DPCM: Linear Prediction Filter



$$\epsilon = E((x(n) - \hat{x}(n))^2);$$

prediction error

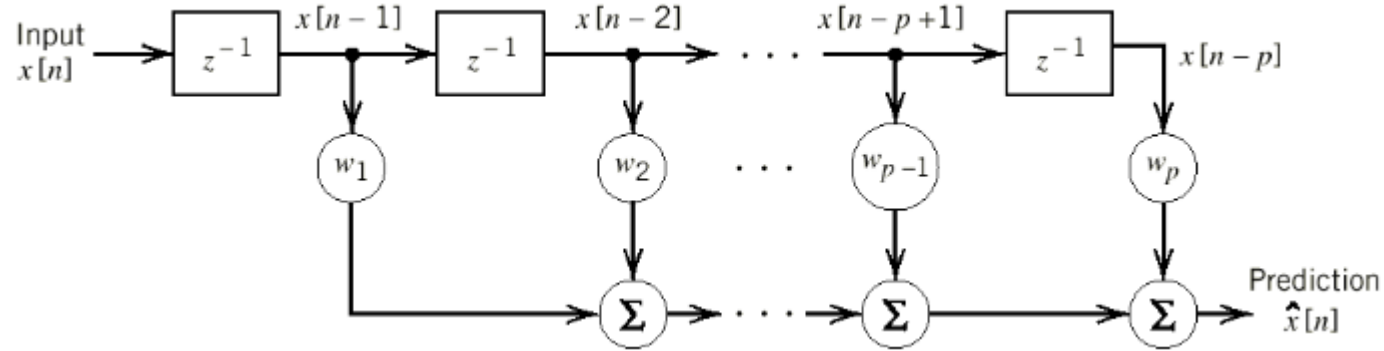
Substituting $\hat{x}(n) = w_1x(n-1) + w_2x(n-2) + \dots + w_px(n-p)$, the prediction error becomes:

$$\epsilon = E((x(n) - w_1x(n-1) + w_2x(n-2) + \dots + w_px(n-p))^2)$$

Expanding ϵ and taking expectation of all terms, we get:

$$\epsilon = E(x(n)^2) - 2 \sum_{i=1}^p w_i E[x(n)x(n-i)] + \sum_{i=1}^p \sum_{j=1}^p w_i w_j E[x(n-i)x(n-j)]$$

DPCM: Linear Prediction Filter



Recognize that: $R_x(i) = E[x(n)x(n-i)]$ is the autocorrelation function of $x(t)$.

Differentiating ϵ with respect to w_i , setting the derivative to zero, and solving, we get (assuming $p=3$)

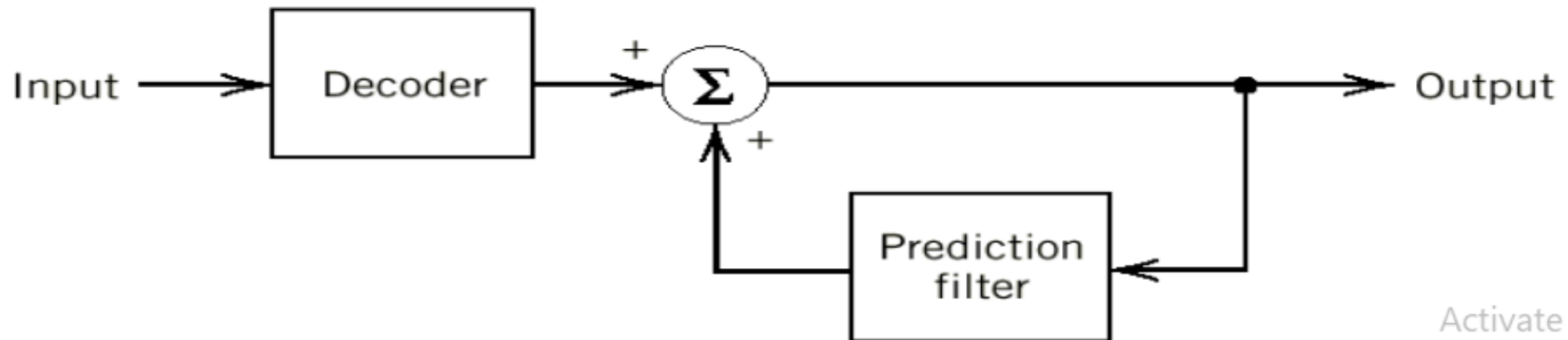
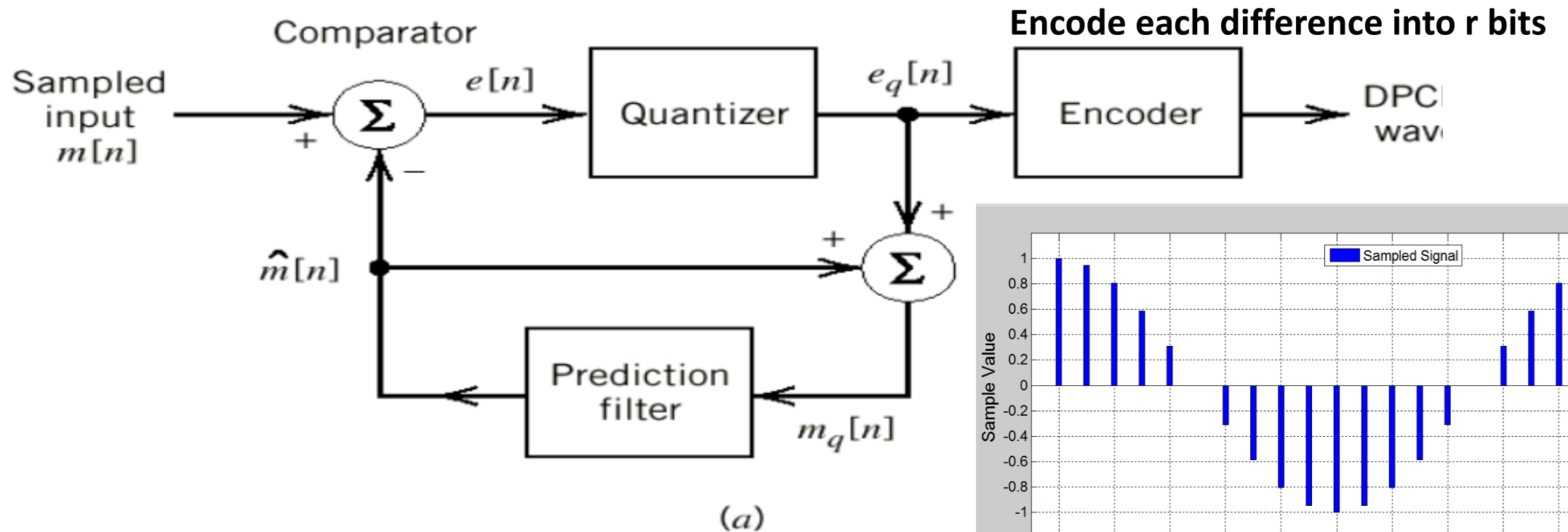
$$\begin{bmatrix} R_x(0) & R_x(1) & R_x(2) \\ R_x(-1) & R_x(0) & R_x(1) \\ R_x(-2) & R_x(-1) & R_x(0) \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} R_x(1) \\ R_x(2) \\ R_x(3) \end{bmatrix} \quad \begin{matrix} R(1) = R(T_s) \\ R(2) = R(2T_s) \end{matrix}$$

If $p=1$, the above equation reduces to

$$w_1 = R_x(1) / R_x(0)$$

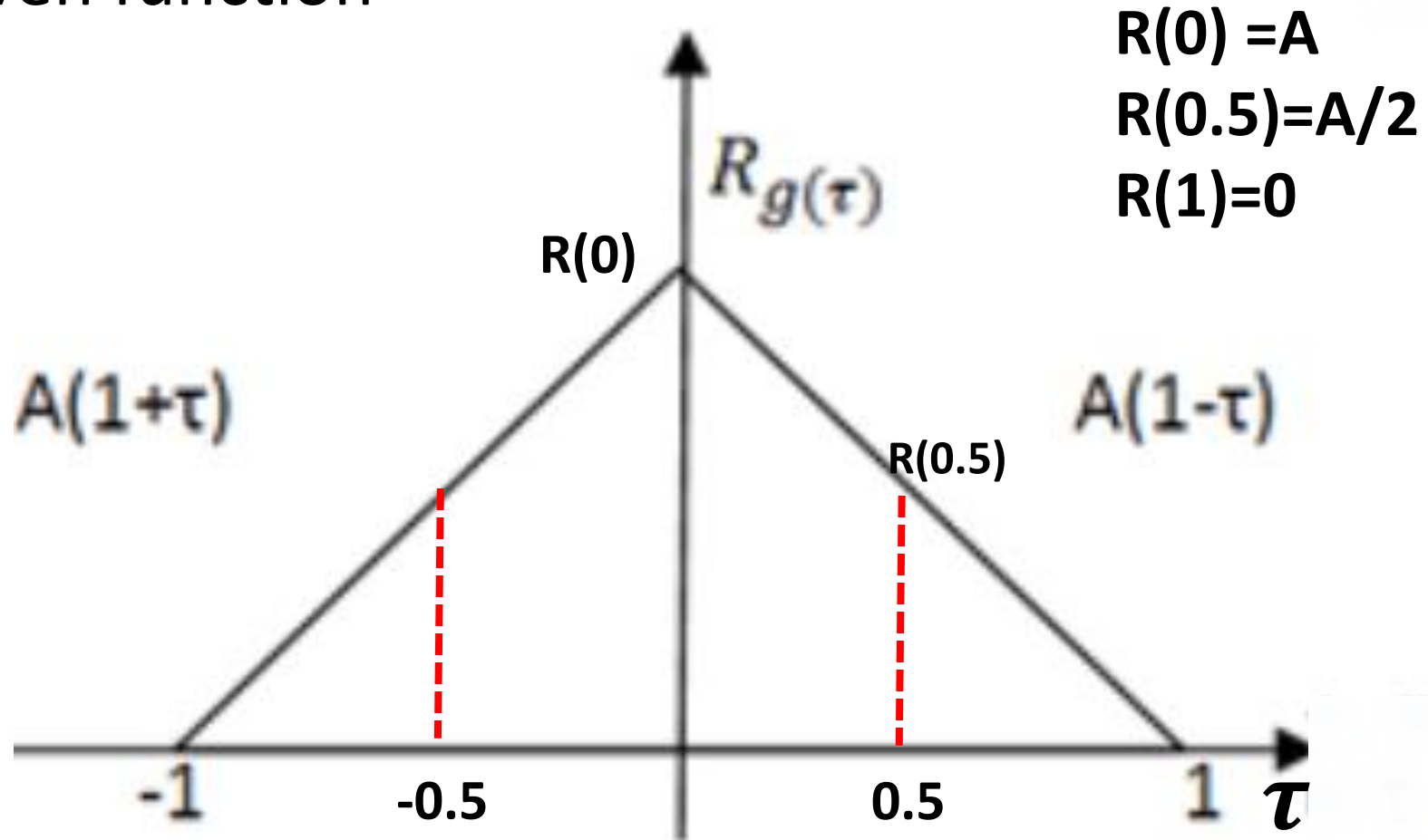
Note that: $R_x(-1) = R_x(1), R_x(-2) = R_x(2), R_x(1) = R_x(T_s), R_x(2) = R_x(2T_s), R_x(3) = R_x(3T_s)$.

DPDM: Transmitter and Receiver

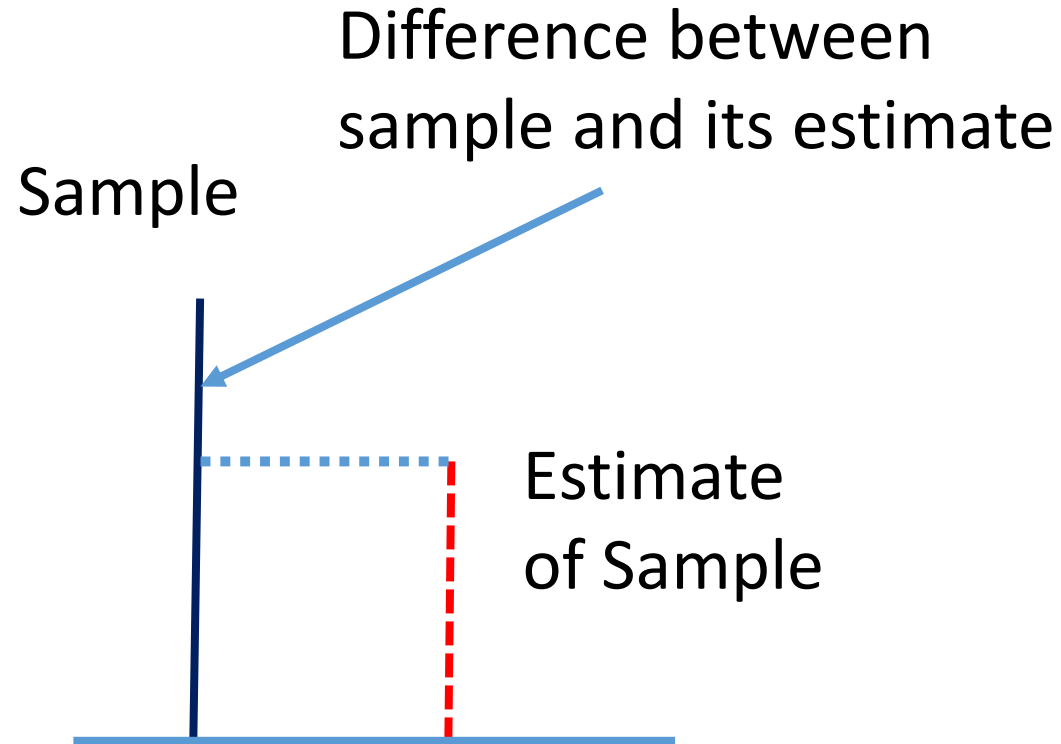


DPCM: Autocorrelation Function

R is an even function



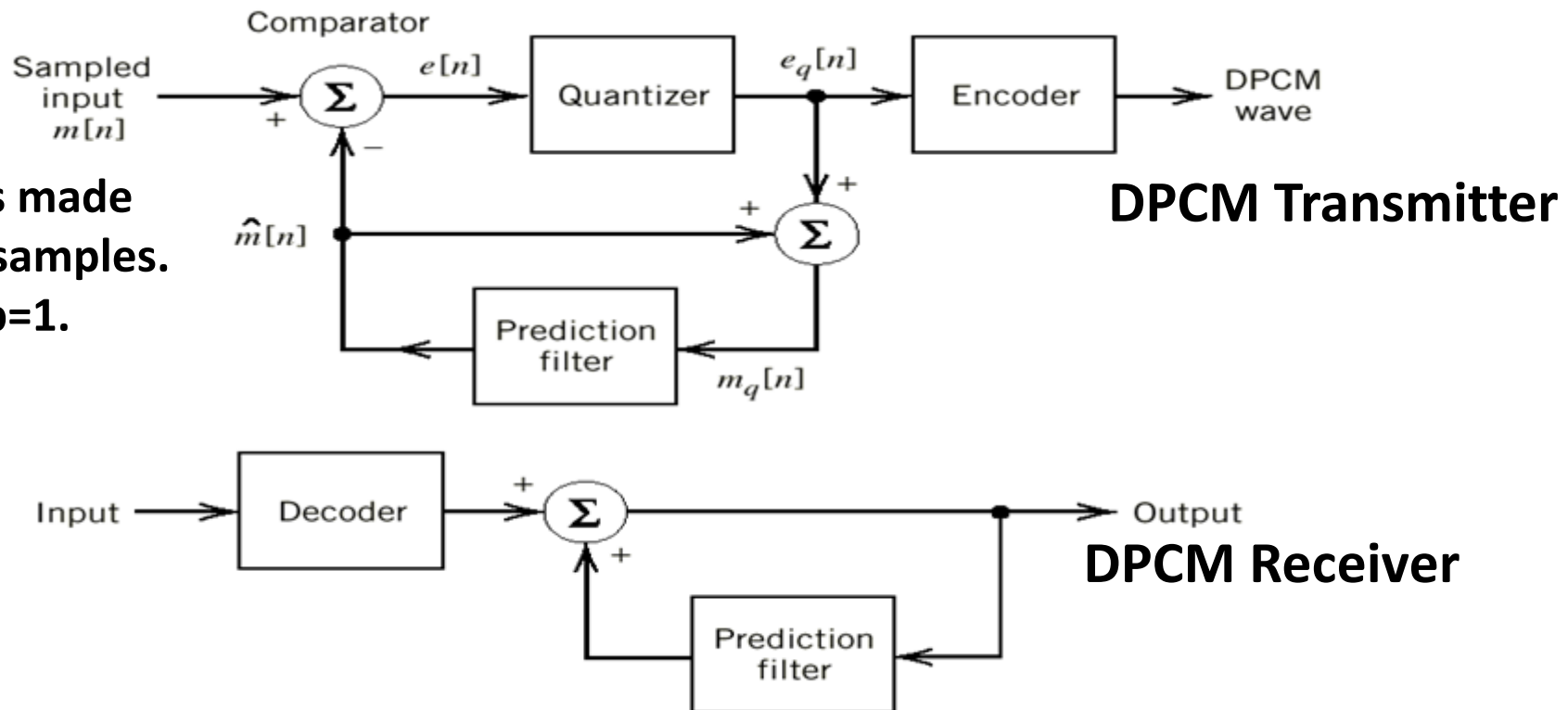
DPCM: Concluding Summary



- **At transmitter:**
 - Samples are known
 - Estimate is known since estimate is a linear function of the samples.
- Transmit
- **Difference = Sample – Estimate**
- **At receiver:**
 - Receive Difference
 - Construct Estimate
- **Sample = Estimate + Difference**

Delta Modulation

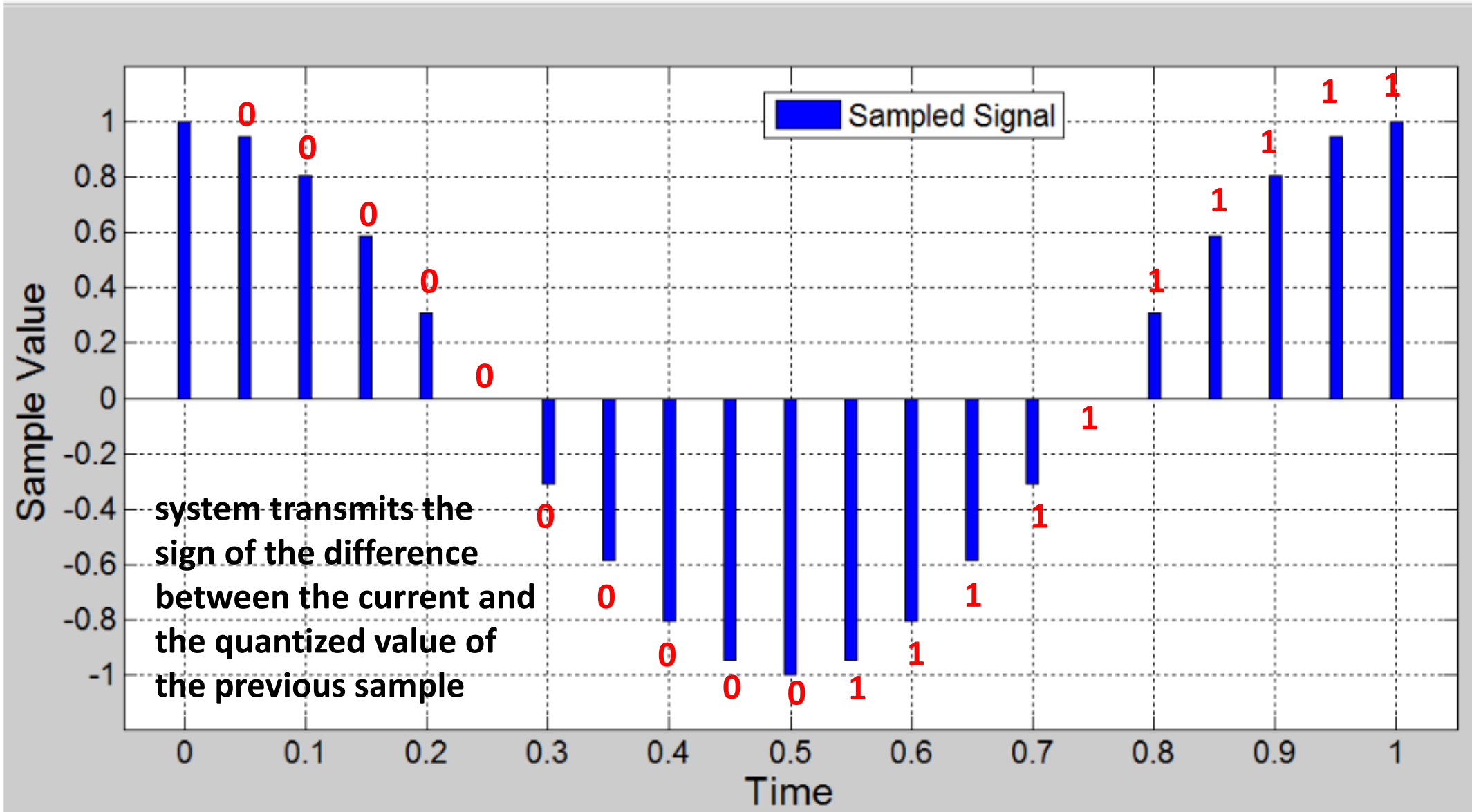
- **Remark:** Before you attend this lecture, please attend the previous one on DPCM.
- Delta modulation (DM), is a special case of Differential Pulse Code Modulation (DPCM).
- The order of the prediction filter in delta modulation is **p=1** and **represents only the quantized value of the previous sample**. The **number of quantization levels is two**.
- In this scheme, the system transmits the sign of the difference between the current sample and the quantized value of the previous sample. The **sign is represented by a single bit**.



Prediction in DPCM is made based on p previous samples. In delta modulation p=1.

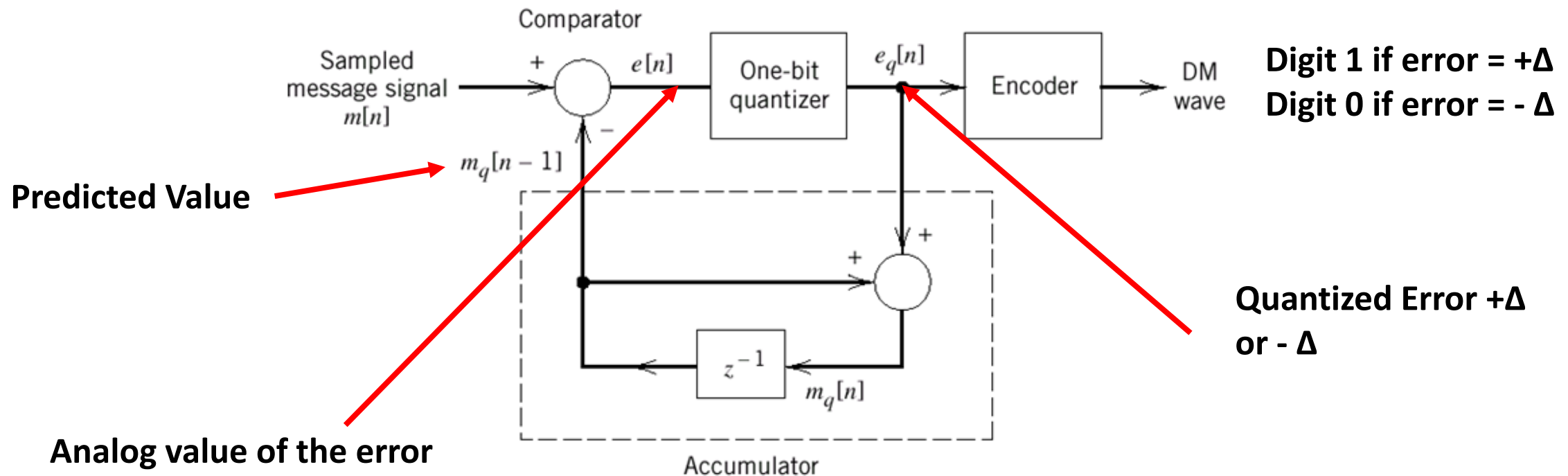


Delta Modulation: Basic Idea



Delta Modulation: The Transmitter Side

- The transmitter side consists of the comparator, the one bit quantizer, the encoder, and the accumulator.
- The accumulator (an integrator) adds the new quantized difference ($+\Delta$ or $-\Delta$) to the old predicted value to generate the new predicted value.
- The output of the predictor is a staircase approximation of the message signal.



Delta Modulation: The Transmitter Side

Let $m[n] = m(nT_s)$, $n = 0, \pm 1, \pm 2, \dots$

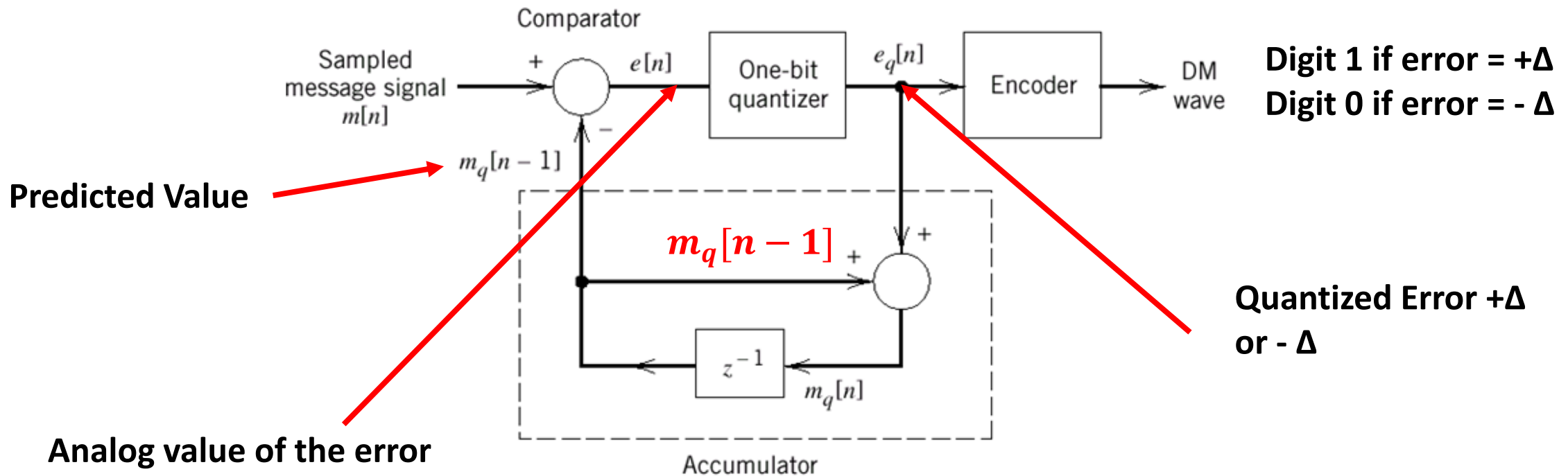
where T_s is the sampling period and $m(nT_s)$ is a sample of $m(t)$. The error signal is

$$e[n] = m[n] - m_q[n-1]$$

$$e_q[n] = \Delta \operatorname{sgn}(e[n]); \text{ quantized error}$$

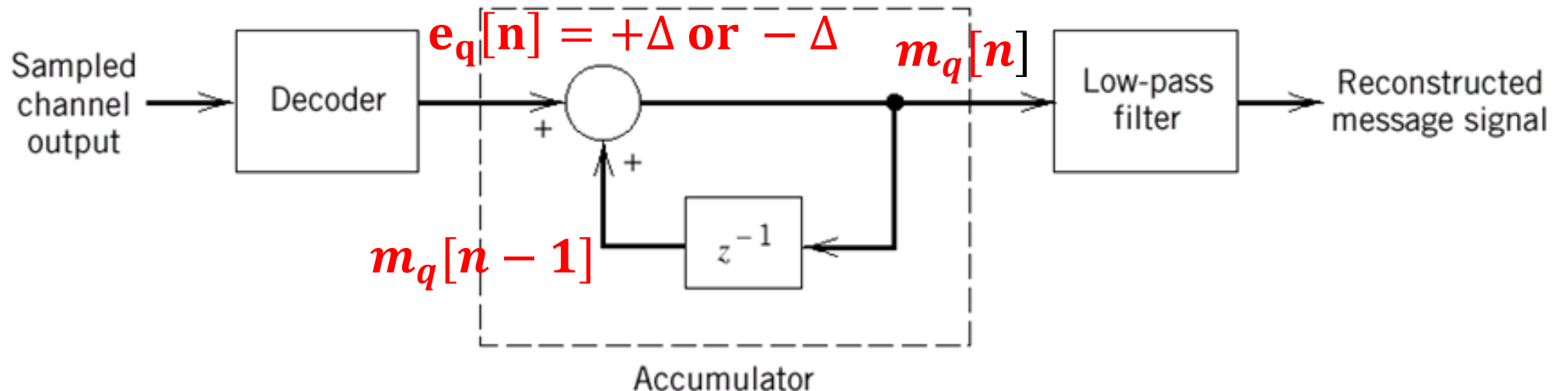
$$m_q[n] = m_q[n-1] + e_q[n]$$

where $m_q[n]$ is the quantizer output, $e_q[n]$ is the quantized version of $e[n]$, and Δ is the step size



Delta Modulation: The Receiver Part

- The receiver part consists of the decoder, the accumulator, and a low pass filter.
- The decoder interprets a zero as $-\Delta$ and one as $+\Delta$. These deltas represent the differences between current and previous samples.
- The accumulator regenerates the predicted staircase signal.
- The low pass filter smoothens the predicted signal by removing high frequency components.
- The reconstructed signal $m_q(t)$ is the same as the predicted signal used at the transmitter side

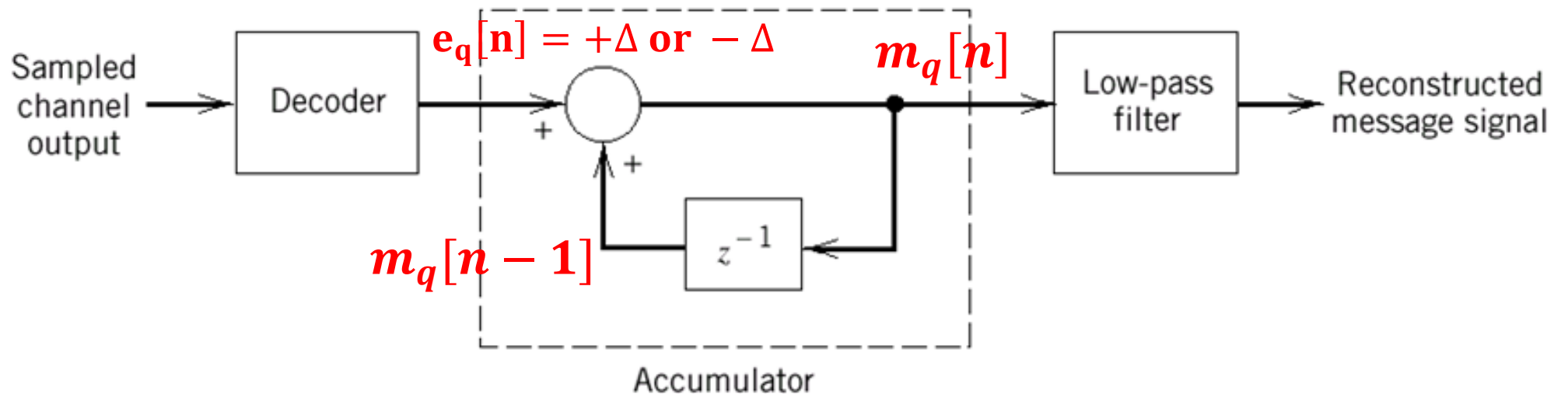


Delta Modulation: The Receiver Part

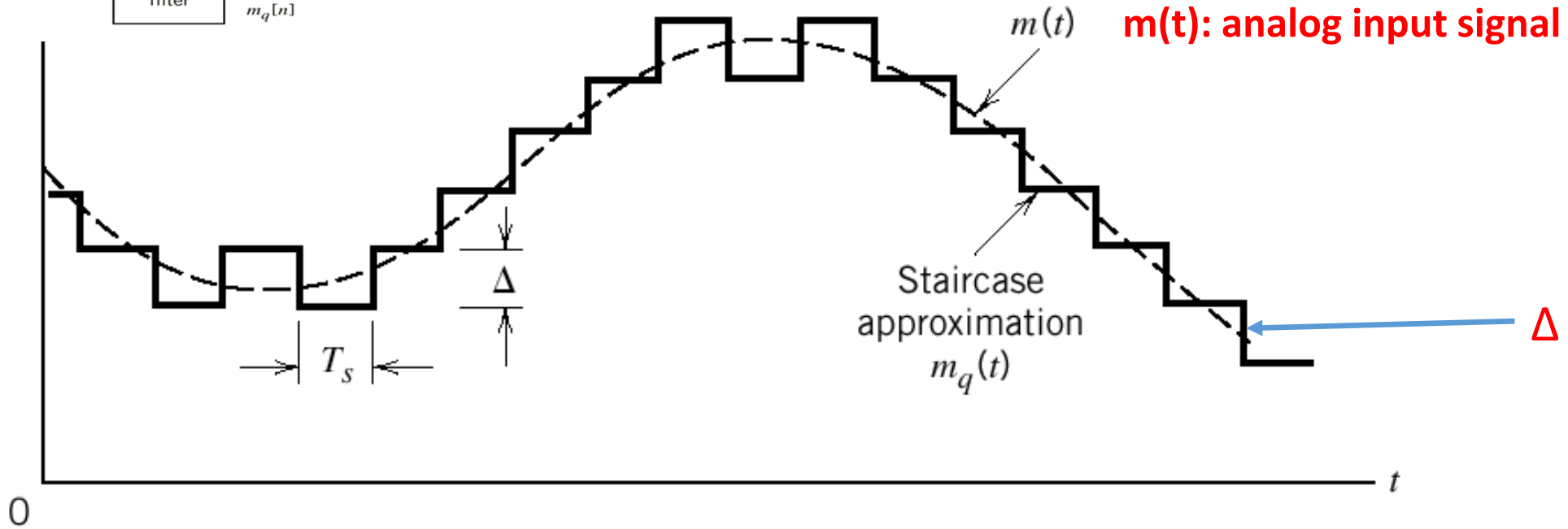
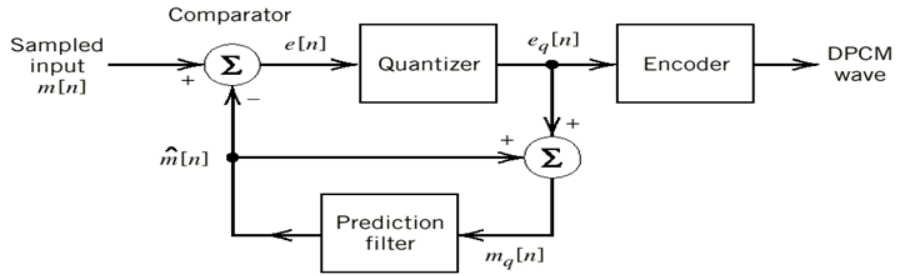
$$m_q[n] = m_q[n-1] + e_q[n]$$

$$\Rightarrow m_q[n] = \Delta \sum_{i=1}^n \text{sgn}(e[i])$$

$$= \sum_{i=1}^n e_q[i]$$



Delta Modulation: Basic Operation



(a)

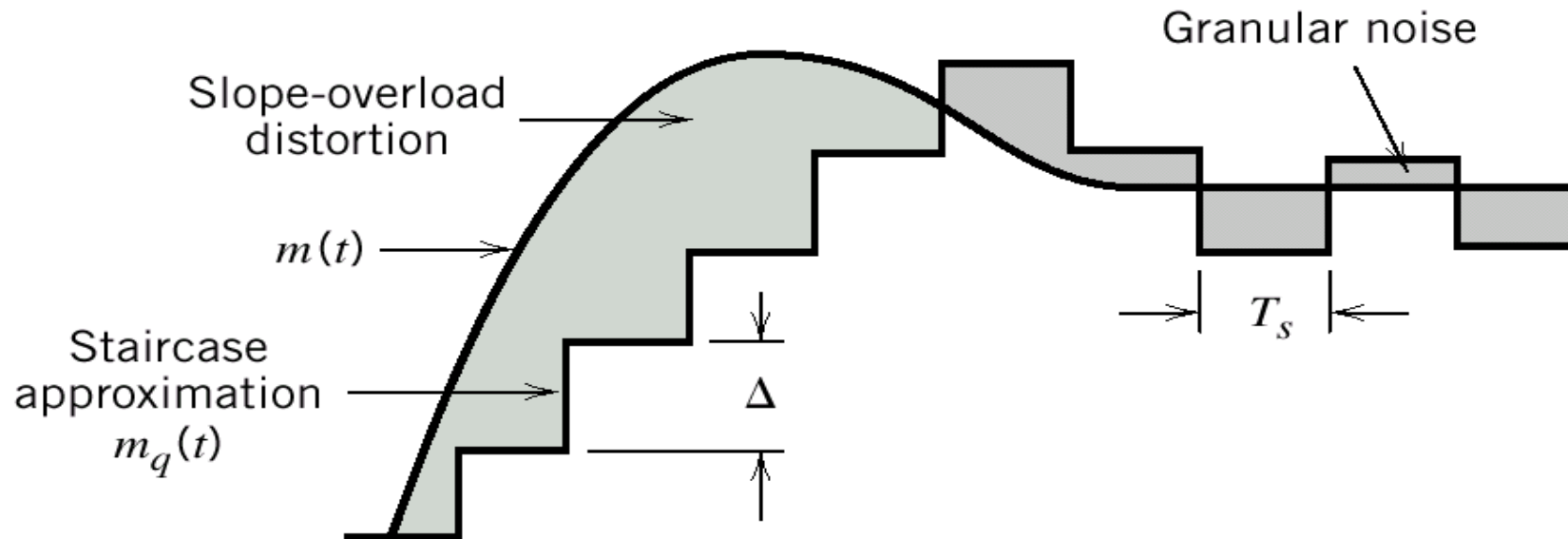
Binary sequence at modulator output

0 0 1 0 1 1 1 1 1 0 1 0 0 0 0 0 0



Slope Overload Distortion and Granular Noise

- **Slope overload** distortion is due to the fact that the staircase approximation $m_q(t)$ can't follow closely the actual curve of the message signal $m(t)$. In contrast to slope-overload distortion, **granular noise** occurs when Δ is too large relative to the local slope characteristics of $m(t)$. granular noise is similar to quantization noise in PCM.
- It seems that a large Δ is needed for rapid variations of $m(t)$ to reduce the slope-overload distortion and a small Δ is needed for slowly varying $m(t)$ to reduce the granular noise. The optimum Δ can only be a compromise between the two cases.
- To satisfy both cases, an adaptive DM is needed, where the step size Δ can be adjusted in accordance with the input signal $m(t)$ (not to be covered in this lecture)



Slope Overload

Slope overload occurs when the signal changes at a rate faster than that of the predicted signal. To avoid slope overload, we must have

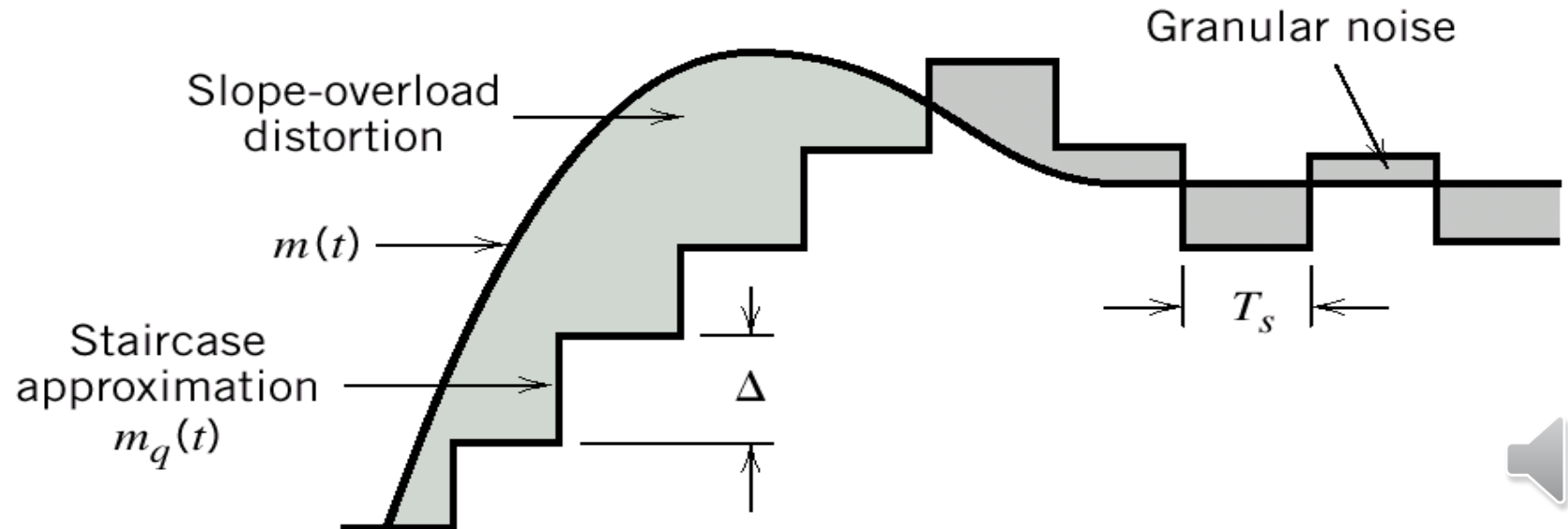
$$\frac{\Delta}{T_s} \geq \max \left| \frac{dm(t)}{dt} \right|$$

When $m(t) = A_m \cos(2\pi f_m t)$, the condition for avoiding slope overload becomes

$$\frac{\Delta}{T_s} \geq 2\pi A_m f_m ; \text{ OR } \Delta \geq 2\pi T_s A_m f_m$$

As we can see, slope overload depends on three factors:

- Sampling frequency (larger sampling, reduces the effect)
- Message amplitude (larger amplitude, increases the effect)
- Message frequency (larger message frequency, increases the effect)



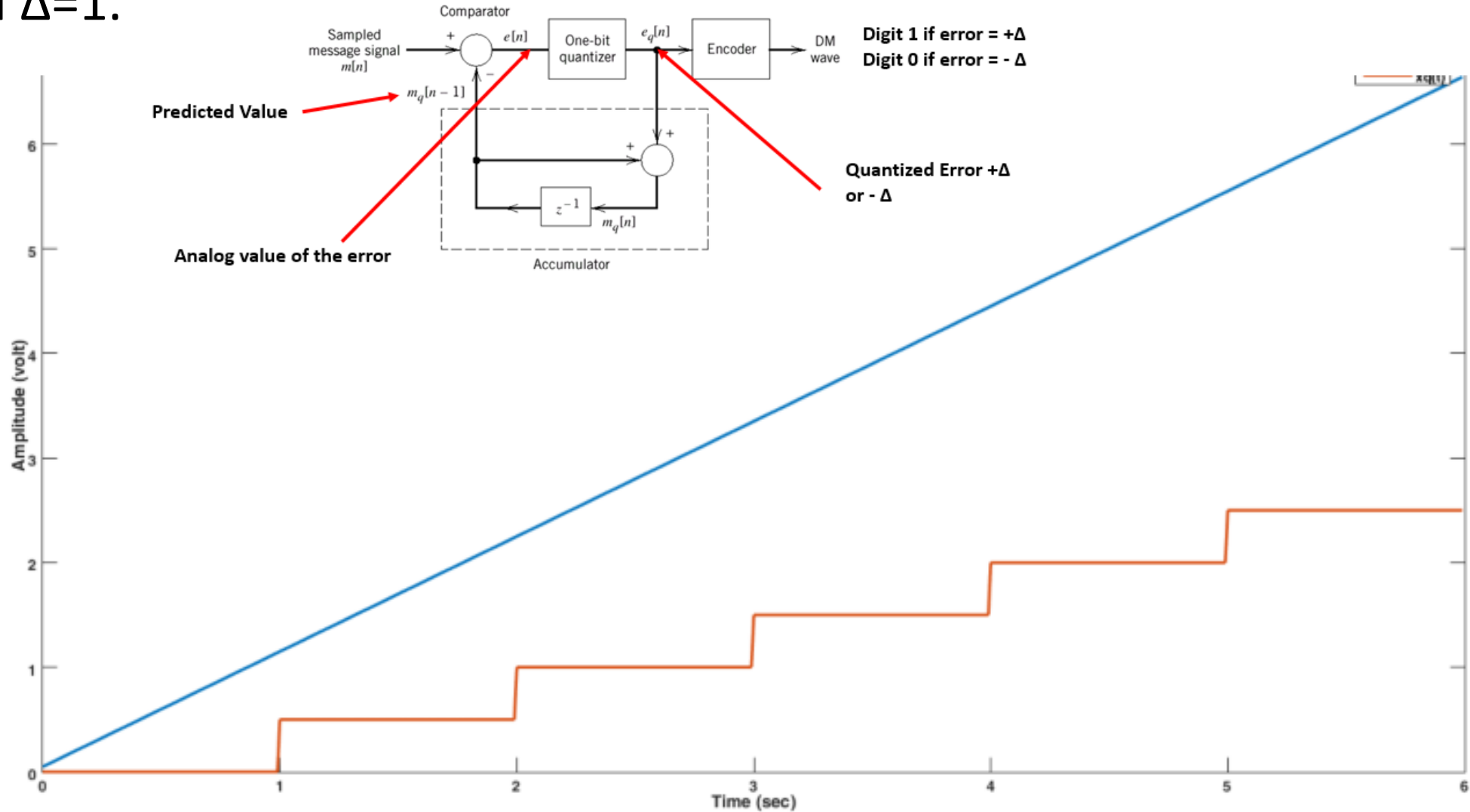
Adaptive Delta Modulation

- The step size in delta modulation affects the quality of the transmitted waveform (slope overload or granular noise).
- A larger step-size is needed in the steep slope of modulating signal
- a smaller step size is needed where the message has a small slope
- In adaptive delta modulation, the step size is adjusted via a feedback control signal so as to reduce both slope overload and granular noise effects.
- ADM quantizes the difference between the value of the current sample and the predicted value of the next sample. It uses a variable step height to predict the next values, for the faithful reproduction of the fast varying values.



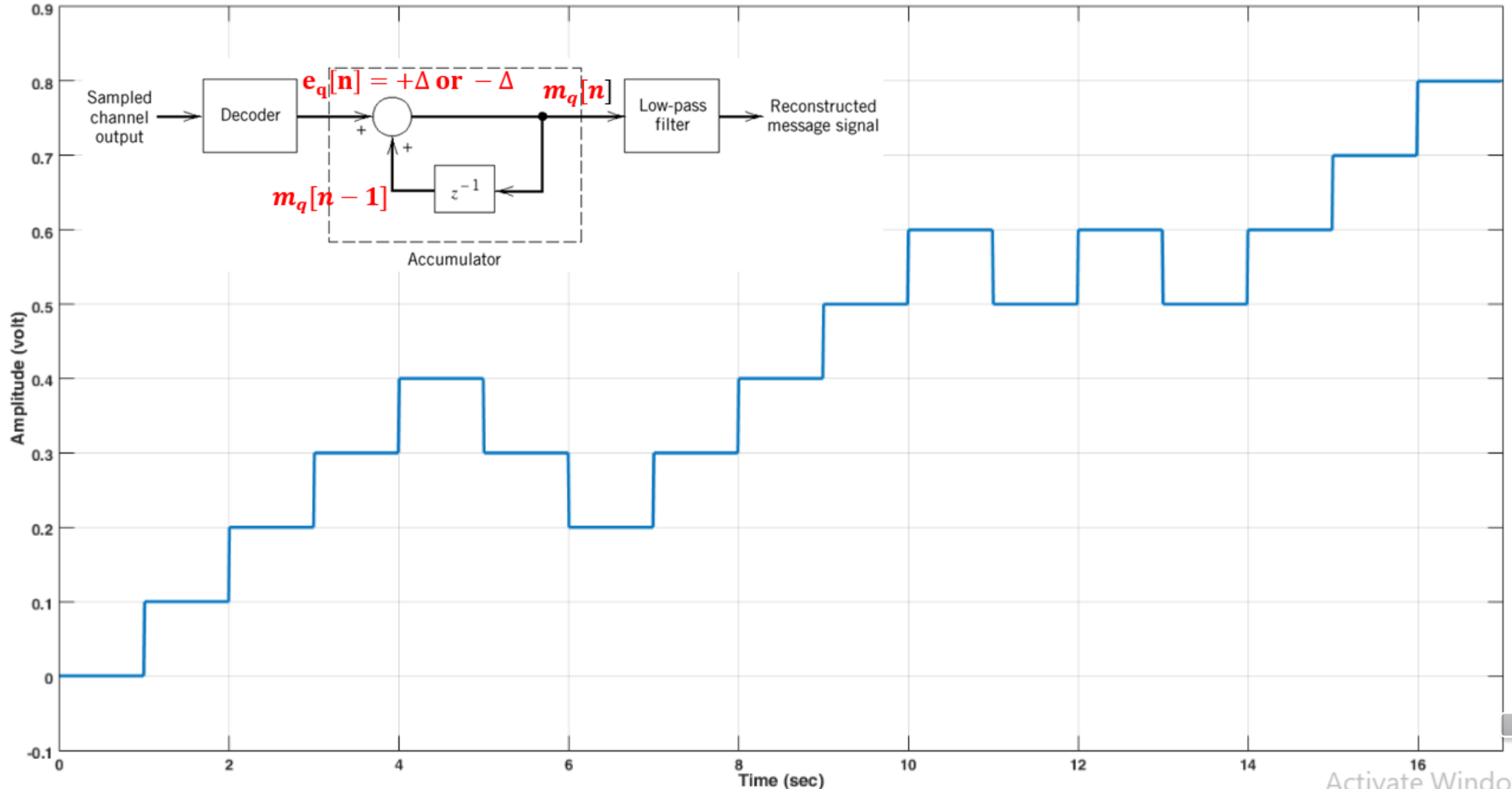
Delta Modulation: Example

- Draw the output of the DM given that the input corresponds to $x(t) = 1.1t + 0.05$ when the input is sampled at $t = 0, 1, 2, 3, 4, 5, \dots$ and $\Delta=1$.



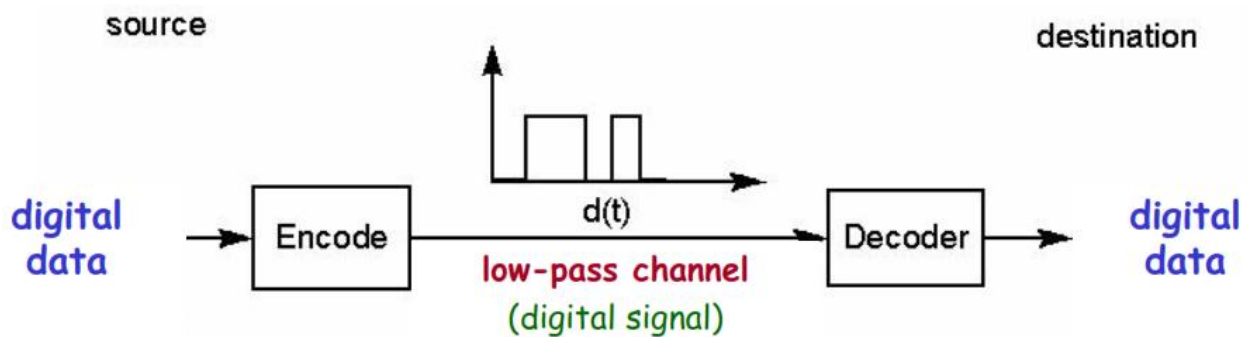
Delta Demodulation: Example

- Reconstruct a staircase signal at the receiver side of a delta demodulator with $\Delta = 0.1V$, when the received data sequence is 1 1 1 1 0 0 1 1 1 1 0 1 0 1 1 1.



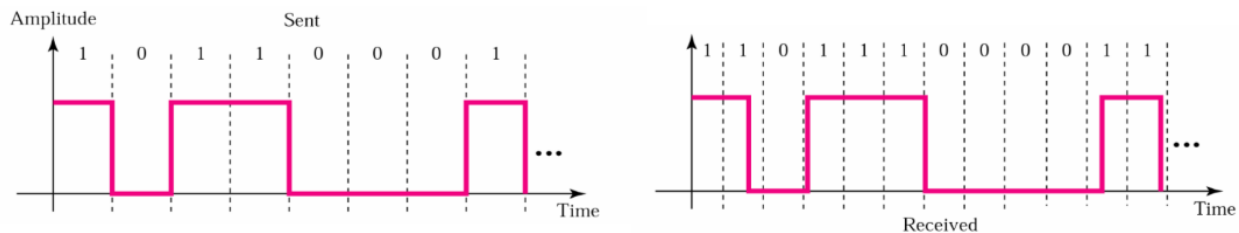
Line Encoding

- The assignment of pulses (an electrical signal) to the binary digits that come out of the PCM or DPCM system.
- Line coding encodes the bit stream for transmission through a line, or a cable.
- It is used for communications between the CPU and peripherals, and for short-distance baseband communications, such as the Ethernet.



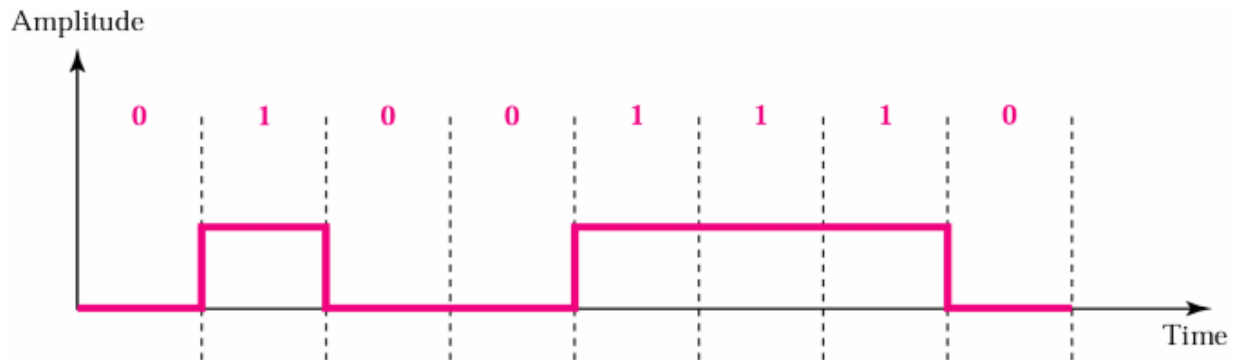
Two Design Considerations

- **DC Component in Line Coding:** Some line coding schemes have a residual (DC) component, which is generally undesirable.
 - Transformers do not allow passage of DC component.
 - DC component \Rightarrow extra energy – useless!
- **Self-Synchronization (clocking):** To correctly interpret signal received from sender, receiver's bit interval must exactly correspond to sender's bit intervals
 - If receiver clock is faster/slower, bit intervals not matched \Rightarrow receiver misinterprets signal
 - Self-synchronizing digital signals include timing information in itself, to indicate the beginning & end of each pulse



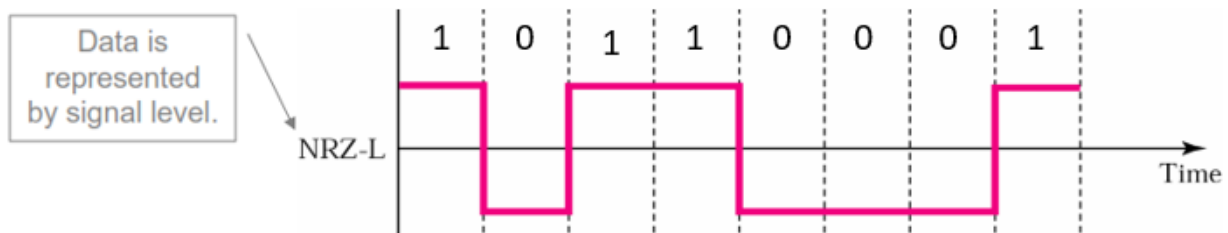
Unipolar Line Coding (Unipolar non-return to zero)

- Uses only one non-zero and one zero voltage level to represent binary digits 1 and 0
- Simple to implement, but obsolete due to two main problems:
 - Presence of a DC component.
 - Lack of synchronization for long series of 1-s or 0-s



Polar Line Coding: Polar non-return to zero

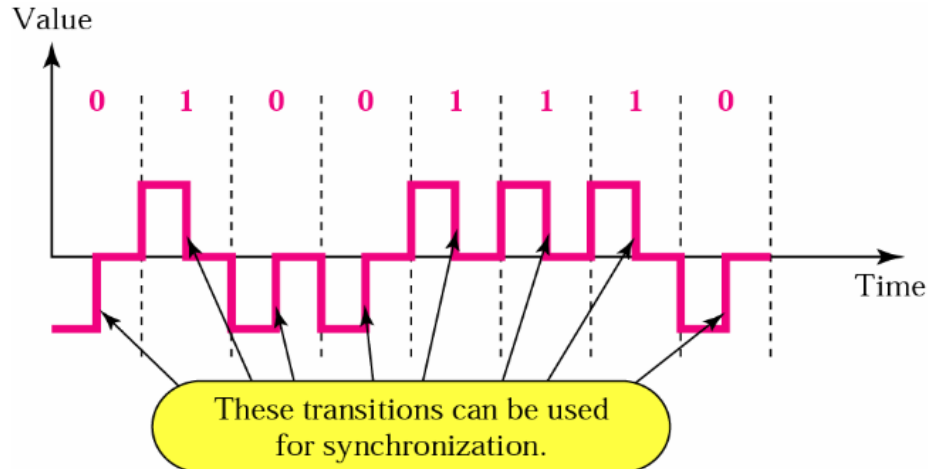
- Uses two non-zero voltage level to represent digits 1 and digit 0. +ve for 1 and -ve for 0
- No DC component is present
- Poor synchronization for long series of 1-s or 0-s



Polar Non-return to Zero

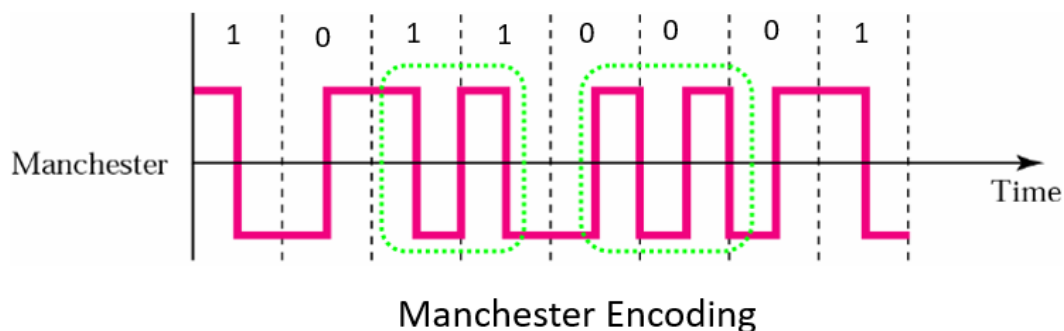
Polar Line Coding: Polar return to zero

- Uses two non-zero voltage level to represent digits 1 and digit 0. +ve for 1 and -ve for 0. **Must** return to zero halfway through each bit interval.
- No DC component is present.
- Perfect synchronization for long series of 1-s or 0-s.
- Twice the bandwidth required for polar non-return to zero, $B.W \propto \frac{1}{pulse\ width}$.



Manchester Line Coding

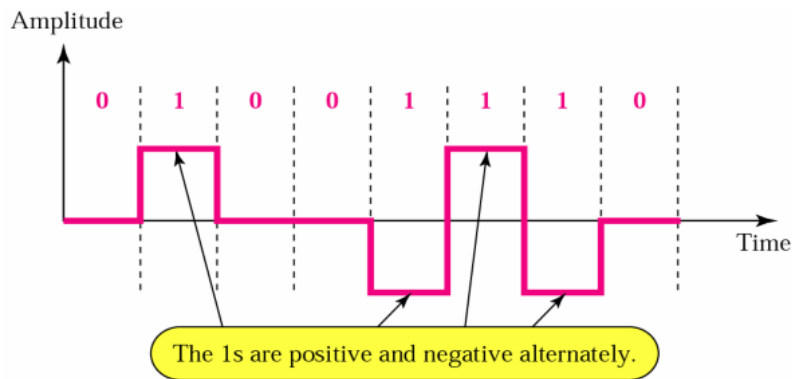
- Inversion at the middle of each bit interval is used for both synchronization and bit representation
- Digit 0 \Rightarrow neg-to-pos transition, 1 \Rightarrow pos-to-neg transition
- Perfect synchronization for long series of 1-s or 0-s
- There is always transition at the middle of the bit, and maybe one transition at the end of each bit.
- Fine for alternating sequences of bits (10101), but wastes bandwidth for long runs of 1-s or 0-s.
- Used by IEEE 802.3 (Ethernet).
- No DC component is present.
- Twice the bandwidth required for polar non-return to zero. Two pulses are used to represent one bit.



Bipolar Line Coding

- Uses two non-zero and zero voltage level for representation of two data levels.
- **0 = zero level; 1 = alternating positive and negative level.**

- If first bit 1 is represented by a positive amplitude, second will be represented by negative amplitude, third by positive, etc.
- Less bandwidth required than with Manchester coding (for any sequence of bits).
- Loss of synchronization is possible for long runs of 0-s.
- No DC component is present.



2B1Q (2 Bipolar to 1Quaternary) Line Coding

- Data patterns of size 2 bits are encoded as one signal element belonging to a four-level signal.
- Data is sent two time faster than with polar non-return to zero.
- Receiver has to discern 4 different thresholds

Binary Input	Output Voltage
00	-3
01	-1
10	1
11	3

